Strategic Asset Allocation  
with Relative Performance Concerns*

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Abstract

This article analyzes the dynamic portfolio choice implications of strategic interaction among money managers. The strategic interaction is modeled as managers’ having relative performance concerns in their objectives, either due to money flows or behavioral considerations. We provide tractable formulations of relative performance concerns between two risk-averse managers in a continuous-time setting, and solve for their equilibrium policies in closed-form. Under a formulation with relative performance concerns smoothly affecting the managers at all levels of wealth, we obtain a unique Nash equilibrium. We demonstrate that much of the novel investment behavior depends on whether a manager is a chaser, increasing investments in response to the other’s increasing hers, or a contrarian. The managers are chasers for empirically plausible parameters and increase their optimal risk exposures due to the presence of strategic interaction. We then consider a formulation with asymmetric relative performance concerns, where a manager gets money flows, and hence displays relative concerns, only if her relative return is above a performance threshold. Such relative concerns induce the manager to engage in risk shifting around the threshold, so as to either end up as a winner (getting money flows) or a loser (no flows). Here, we do not always obtain a Nash equilibrium since the managers cannot agree on the winner. For sufficiently similar managers, we obtain equilibria, but multiple, since managers care only about the total number of winning states. We recover a unique equilibrium, however, with a sufficiently high threshold, where the managers’ risk aversion prevents them from taking huge gambles, leading both to be losers around the threshold.

JEL Classifications: G11, G20, D81, C73, C61.
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1. Introduction

This paper analyzes the portfolio strategies of money managers in the presence of strategic interactions, arising from each manager’s desire to perform well relative to the other managers. There are several reasons why managers may care about relative performance. First, given the prevalent finding in money management that top performing mutual and hedge funds attract more new money than their less successful peers (e.g., Chevalier and Ellison (1997), Sirri and Tuffano (1998) for mutual funds, Agarwal, Daniel, and Naik (2004) for hedge funds), a fund manager has incentives to outperform the peers so as to increase her assets under management, and hence, her compensation.\footnote{A standard explanation for a positive relation between money flows and relative performance is that investors respond to widely published fund rankings (MorningStars, Business Week, Forbes, Institutional Investor) when choosing which fund to invest into.} Second, relative concerns may arise due to psychological aspects of human behavior, such as envy or crave for higher social status. Traditional finance has long ignored these features, assuming that investors with such “irrational” traits cannot have a lasting impact on financial markets. However, it is now well recognized that relative considerations play an important role in explaining various phenomena, the most notable example being the success of “keeping-up-with-the-Joneses” models (Abel (1990)).

When discussing the interaction of mutual funds in the presence of relative performance concerns, Brown, Harlow, and Starks (1996) appeal to the notion of a tournament in which mutual funds are the competitors and money flows are the prizes awarded based on relative ranking. Since then, a rapidly growing literature (Chevalier and Ellison (1997), Busse (2001), Qiu (2003), Goriaev, Nijman, and Werker (2005), Reed and Wu (2005)) address the tournament hypothesis by looking at how risk taking behavior responds to relative performance. However, without an underlying theory, it is not clear whether a particular finding is consistent with a hypothesis, or on the contrary, is a puzzling result that needs to be further investigated. As demonstrated by recent theoretical work (Basak, Pavlova, and Shapiro (2007), Chen and Pennacchi (2005), Makarov (2006)), relying on “intuitive” reasoning or on insights obtained in other tournament settings (e.g., Lazear and Rozen (1981) for labor market tournaments, Ehrenberg and Bognanno (1990) for sport competitions) may well lead to inaccurate predictions regarding funds’ risk taking incentives. The effects of strategic considerations are likely to be the strongest when there is a small number of funds competing against each other. A natural real-life case is when several top-performing funds compete for the leadership, and so we adopt this interpretation in our subsequent discussions.

Our objective in this paper is to investigate the dynamic strategic interaction of money managers with relative performance concerns, and to analyze their ensuing investment policies. While several recent studies attempt to analyze mutual fund tournaments, their results are obtained under fairly specialized economic settings (as discussed below). To our best knowledge, ours is the first comprehensive analysis of the portfolio choice effects of strategic interactions within a workhorse dynamic asset allocation framework, allowing us to derive a rich set of implications. We consider two risk averse money managers, interpreted as either mutual fund managers, hedge
fund managers, or simply traders. We adopt the familiar Black and Scholes (1973) continuous-time economy for investment opportunities, and assume constant relative risk aversion (CRRA) preferences for a normal manager with no relative performance concerns. We model relative performance concerns by postulating that a manager’s objective function depends (positively) on the ratio of her horizon investment return over the other manager’s return, in addition to her own horizon wealth. Our leading interpretation for the presence of these relative performance concerns is based on money flows, which capture the desire of a money manager to attract new money by outperforming the other manager—we formally justify this in our analysis.

We study two objective functions of relative performance concerns. The first is a smooth specification, governed by a relative performance bias parameter, affecting the managers at all wealth levels, and converging to the case of a normal manager with no relative concerns for a zero bias parameter. Such a specification is justified by a smooth money flow-performance relation, where the bias is driven by the money flow elasticity. Additionally, the presence of flows changes a manager’s attitude towards risk, increasing it for risk aversion greater than unity, and decreasing it otherwise. The second specification, justified by an option-like flow-performance relation, has relative performance concerns enter the objectives asymmetrically: if the manager’s performance exceeds a performance threshold, her objectives are as in the smooth specification (getting money flows), and otherwise if she performs relatively poorly, her objectives are as for a normal manager with no relative concerns (getting no flows). The importance of this specification is that it exhibits local convexities around the performance threshold, and hence induces a manager to engage in risk shifting, or gamble, as widely discussed in the literature in the context of fund management or executive compensation (Chevalier and Ellison (1997), Carpenter (2000)). In characterizing managers’ behavior, we appeal to the pure-strategy Nash equilibrium concept, in which each manager strategically accounts for the dynamic investment policies of the other manager, and the equilibrium policies of the two managers are mutually consistent.

Under our smooth specification in which relative concerns affect each manager at all levels of her relative performance, we demonstrate that an equilibrium always obtains and provide explicit solutions to the equilibrium investment policies. We show that an important feature of the optimal behavior of a manager with relative performance concerns is whether she is “chasing” the other manager, increasing her optimal policy in response to the other manager’s increasing hers, or is “contrarian” to the other manager, decreasing her optimal policy in response to the other’s increasing hers. In our formulation, a manager is a chaser if her risk aversion coefficient is greater than unity (more empirically plausible), and is a contrarian otherwise. In particular, the presence of relative performance concerns induces a chaser to increase her optimal risk taking as compared to her normal policy, and a contrarian to decrease her risk taking. Consequently, when the two managers are behaving strategically and are chasers, they both increase their Nash equilibrium risk exposures in response to an increased relative performance concern by any

\[ \text{In addition to the money management industry, our analysis can also be applied to study the behavior of traders working in the same investment bank. Indeed, while it may not be explicitly written in a contract, it is common knowledge that promotion of traders much depends on their relative (to colleagues) success. Hence, a trader concerned about her career is likely to have relative performance considerations as one of her objectives.} \]
manager. This is because a more pronounced concern makes a manager chase the other manager even stronger, which is achieved by increasing her risk exposure. The other manager, also a chaser, responds by increasing her risk exposure – the managers’ actions effectively reinforce each other. As to be expected, because the managers interact with each other, each manager’s optimal policy depends not only on her own but also on the opponent’s risk aversion. If one manager’s risk aversion is increased, she reduces her risk exposure, and the other manager, a chaser, responds by reducing hers. Hence, in a new equilibrium the two chasers both reduce their equilibrium risk exposures. We show that these effects on managers’ risk taking behavior get confounded when we explicitly account for the flow-performance relation. In particular, a higher flow elasticity increases the manager’s relative performance bias, while also increasing a chaser’s attitude towards risk. Consequently, while the more risk averse chaser increases her risk exposure, the more risk tolerant manager reduces hers.

We next analyze the managers’ strategic asset allocations under the asymmetric relative performance specification, where a manager only gets money flows if her relative return is above a performance threshold. Here, we demonstrate that an additional critical feature in the managers’ strategic interaction is that a manager only chooses two outcomes: “winning” by outperforming the other manager, or “losing” by underperforming and getting no flows, and never opting for a “draw”. This is due to the local convexity around the performance threshold, inducing the manager to gamble so as to end up either a winner or a loser, thus avoiding a draw. However, this risk shifting also leads to the potential non-existence of a Nash equilibrium, since when both managers’ performances are close to their thresholds (in the convex region), they cannot agree on who the winner is. Such a situation occurs when the managers’ attitude towards risk are considerably different, where one manager may want to outperform the other by just a little to become a winner, while the other manager wants to underperform by a lot to be content with being a loser. If on the other hand, the managers’ risk aversions are sufficiently similar, an equilibrium obtains, with one manager emerging as a winner and the other as a loser. However, since the managers are very similar, the opposite equilibrium is also likely to occur, where the earlier winner and loser positions are switched. We show that both of these non-existence and multiplicity issues are resolved when the performance threshold is sufficiently high, leading to a unique Nash equilibrium. This is because with a high threshold, the two managers cannot be close to their thresholds at the same time, which is the main cause for the non-existence and multiplicity. Moreover, when the threshold is high both managers may optimally choose to not compete for money flows, but be losers and follow their normal policies. This unique equilibrium is more likely to exist for higher values of risk aversions or lower money flow elasticities, making the two managers content with being losers as the incentives to gamble are weaker.

We provide a full characterization of the unique equilibrium with the asymmetric relative performance specification. Three possible equilibrium outcomes occur depending on the economic conditions at the horizon. In good states, the more risk averse manager performs worse (consistent with normal behavior) and hence is a loser, while the less risk averse manager with the better performance is a winner, getting the money flows. In bad states, the opposite holds, with the
more risk averse manager emerging as the winner, and the less risk averse as the loser. In intermediate states, around their performance thresholds, both managers are losers, not getting any flows. Consequently, the managers’ equilibrium dynamic investments are stochastic, dependent on economic conditions and the likelihood to receive money flows. In the deep underperformance or overperformance regions, with one manager a loser and the other a winner who receives flows, the properties of the equilibrium investments are similar to the smooth-specification case, driven by whether the managers are chasers or contrarians. However, as the managers’ performances get closer to the threshold, they gamble in equilibrium, giving rise to a hump in their stock investments. The gambling and the associated humps are more pronounced for high flow elasticities, since now the “prize” for becoming a winner increases.

Our paper is related to several strands of literature. First is the literature on the effects of strategic considerations on portfolio managers’ choices. Within single- or two-period settings, Goriaev, Palomino and Prat (2003) and Taylor (2003) focus on the risk taking incentives of two risk neutral managers with different levels of interim performance. Palomino (2005) investigates the effect of relative performance objectives on the riskiness of portfolios in a one-period setting where (for the most part) risk neutral fund managers have private information about risky assets and have market power. Our goal is to characterize optimal portfolios in a standard asset allocation setting with risk averse managers, which is not possible under risk-neutrality. Moreover, we find risk aversion to be the critical driving factor in much of our analysis, including chasing/contrarian behavior, risk shifting, existence of equilibrium. We also investigate various possible traits of portfolios, with interim performance being just one, which is shown to not be crucial in our setting. Li and Tiwari (2005), also under risk-neutrality and single/two-period settings, investigate the welfare implications of mutual fund tournaments in the presence of costly information acquisition and find that a tournament can be welfare enhancing. Chiang (1999) investigate how the strategic behavior is affected by the threat of dismissal with risk-averse managers facing two investment choices under two periods. Within a risk-neutral two-period setting, Loranth and Sciubba (2006) look at how the competing funds’ strategies are affected by the threat of a new fund entering a tournament. In a dynamic setting like ours, Browne (2000) investigates a portfolio game between two managers. He primarily focuses on the case when the managers face different financial investment opportunities and have practical objectives (maximizing the probability of beating the other manager, minimizing the expected time of beating the other manager). One of his specifications is similar to a special (knife-edge) case of our smooth specification for relative performance concerns, leading to trivial optimal portfolios if his managers had access to the same assets.

If a peer group consists of a large number of competing funds, strategic interactions are likely to be less pronounced. In this case, the behavior of each fund manager is better described by

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3 Specifically, Browne defines an expected payoff function, common to both managers, which depends on the ratio of manager 1’s wealth to manager 2’s wealth level. The strategic interaction emerges as manager 1 tries to increase this function, while manager 2 to decrease it. From the viewpoint of our analysis, such a specification resembles the case when the two managers are fully biased towards relative concerns, and do not care about absolute performance.
assuming that she seeks to perform well relative to an exogenous benchmark. The manager’s behavior in this case has been recently investigated in Basak, Shapiro, and Tepla (2006), van Binsbergen, Brandt, and Koijen (2007), Cuoco and Kaniel (2005). Several works, including Carpenter (2000), Basak, Pavlova, and Shapiro (2007), have demonstrated that convexities in managers’ objective functions have significant implications for the optimal portfolios, leading to risk shifting behavior. We contribute to this literature by investigating how the risk shifting motives interact with strategic considerations and recover important economic implications, namely the possibility of multiple equilibria or no equilibrium at all – aspects not present in settings without strategic interaction. When these complications are not present and a unique equilibrium exists, our ensuing investment policies are qualitatively similar to those obtained in Basak et al. for a single manager facing a linear-convex flow specification. Their optimal policies, however, are obtained using numerical methods while ours are in closed-form, which greatly facilitates the comparative statics analysis.

Finally, our paper is related to the literature that examines the role of relative wealth concerns in finance. DeMarzo, Kaniel, and Kremer (2007a, 2007b) show that relative wealth concerns may play a role in explaining financial bubbles and excessive real investments. These papers are close in spirit to our work since they also demonstrate how relative wealth concerns may arise endogenously. However, their mechanism for the emergence of relative concerns is a general equilibrium one, and so is notably different from ours. In DeMarzo et al., there is a scarce consumption good whose price increases with the cohort’s wealth, implying that an investor’s relative wealth determines the quantity of the good she can afford. Abel (1990), Gomez, Priestley, and Zapatero (2006), among many others, demonstrate that models with the “catching-up-with-the-Joneses” feature can explain various empirically observed asset pricing phenomena. Goel and Thakor (2005) investigate how envy leads to corporate investment distortions.

Remainder of the paper is organized as follows. Section 2 describes the model and provides the money flows justification for relative performance concerns. Section 3 solves the model under the smooth relative concerns specification, while Section 4 under the asymmetric specification. Section 5 concludes. Proofs are in the Appendix.

2. Economy with Strategic Relative Performance Concerns

2.1. Economic Set-Up

We adopt a familiar dynamic asset allocation framework. We consider a continuous-time, finite horizon \([0, T]\) economy, in which the uncertainty is driven by a Brownian motion \(\omega\). Financial investment opportunities are given by a riskless bond and a risky stock, as in the Black and Scholes (1973) economy. The bond provides a constant interest rate \(r\). The stock price, \(S\), follows a geometric Brownian motion

\[dS_t = \mu S_t dt + \sigma S_t d\omega_t,\]

where the stock mean return, \(\mu\), and volatility, \(\sigma\), are constant.
Each money manager $i$ in this economy dynamically chooses an investment policy $\phi_i$, where $\phi_{it}$ denotes the fraction of fund assets invested in the stock at time $t$, or the risk exposure, given initial assets of $W_{i0}$. The investment wealth process of manager $i$, $W_{it}$, follows
\begin{equation}
\text{d}W_{it} = [r + \phi_{it}(\mu - r)]W_{it}\text{d}t + \phi_{it}\sigma W_{it}\text{d}\omega_t.
\end{equation}

Dynamic market completeness (under no-arbitrage) implies the existence of a unique state price density process, $\xi_t$, with dynamics
\begin{equation}
\text{d}\xi_t = -r\xi_t\text{d}t - \kappa\xi_t\text{d}\omega_t,
\end{equation}
where $\kappa \equiv (\mu - r)/\sigma$ is the constant market price of risk (or Sharpe ratio) in the economy. The state-price density serves as the driving economic state variable in a manager’s dynamic investment problem absent any market imperfections. The quantity $\xi_t(\omega)$ is interpreted as the Arrow-Debreu price per unit probability $P$ of one unit of wealth in state $\omega \in \Omega$ at time $t$. In particular, each manager’s dynamic budget constraint (1) can be restated as (e.g., Karatzas and Shreve (1998))
\begin{equation}
E[\xi_T W_{iT}] = W_{i0}.
\end{equation}

This allows us to equivalently define the set of possible investment policies of managers as being the managers’ horizon wealth, $W_{iT}$, subject to the static budget constraint (2).

2.2. Modeling Relative Performance Concerns

We envision a money manager, interpreted either as a mutual fund manager, a hedge fund manager or a trader, whose investment objective is twofold. First, she seeks to increase the terminal value of her portfolio. This is consistent with maximizing her own compensation given the widespread use of the linear fee structure in the mutual fund industry. Second, the fund manager seeks to perform well relative to a certain peer group comprised of other fund managers. In this paper, we look at the scenario when all fund managers within this peer group have relative performance concerns, and so their investment policies depend on each other. This would not be an interesting question if the peer group were comprised of a large number of funds, in which case the effect of the strategic interactions is likely to be negligible. One could then reasonably assume that an individual manager’s policy does not affect the behavior of other funds. However, when there are few funds competing against each other the strategic interactions may play an important role. We expect this problem to be most pronounced among a small number of the very top funds as they attempt to become year-end top-performers within a peer group.

We postulate that the objective function of manager $i$ has the form
\begin{equation}
v_i(W_{iT}, R_{iT}),
\end{equation}
where $v_i$ is (weakly) increasing in horizon wealth, $W_{iT}$, and horizon relative return, $R_{iT}$. In this baseline analysis, we consider a framework with two fund managers, indexed by $i = 1, 2$. The relative returns of managers 1 and 2, $R_{1T}$ and $R_{2T}$, capture the relative performance concerns and are defined as the ratio of the two managers’ time-$T$ investment returns:
\begin{equation}
R_{1T} = \frac{W_{1T}}{W_{10}} / \frac{W_{2T}}{W_{20}}, \quad R_{2T} = \frac{W_{2T}}{W_{20}} / \frac{W_{1T}}{W_{10}}.
\end{equation}
Without loss of generality, we normalize both managers’ initial assets to be equal, $W_{10} = W_{20}$.

There are several reasons why fund managers may want to take relative performance into consideration. Our leading interpretation is based on fund flows, capturing the desire of money managers to attract new money by outperforming the other manager – we formalize this idea below. Alternatively, the objective specification (3) can be interpreted as capturing the well-known psychological feature that people care about their relative standing in the society or in their profession. There are various ways people may perceive their status, as argued in psychological studies.\(^4\) We elaborate on this when we consider different objective specifications in Sections 3 and 4.

We now provide a rational justification for the objective function (3) by showing that it arises naturally in a setting where managers care directly only about their own wealth. The economic setting is extended as follows. The manager continues to invest beyond date $T$ up until an investment horizon $T'$. Motivated by the empirical research documenting fund-flows to relative performance relationships (Chevalier and Ellison (1997), Sirri and Tufano (1998), Agarwal, Daniel, and Naik (2004)), we further assume that the manager experiences money flows at a rate $f_T$.\(^5\) The flow-performance relationship $f_T$ depends on the manager’s relative performance over the period $[0,T]$, with $f_T > 1$ denoting an inflow and $f_T < 1$ an outflow. The manager’s investment horizon $T'$ (e.g., expected tenure, compensation date), of course, need not coincide with the money flows date $T$ (e.g., quarter- or year-end). So, the objective function (3) is interpreted as the manager’s indirect utility of post-flows horizon wealth.

Manager $i$, $i = 1, 2$, is assumed to have CRRA preferences defined over the overall value of assets under management at time $T' > T$:

$$u_i(W_{T'}) = \left(\frac{W_{T'}}{1 - \bar{\gamma}_i}\right)^{1 - \bar{\gamma}_i}, \quad \bar{\gamma}_i > 0, \bar{\gamma}_i \neq 1. \quad (5)$$

We consider two specifications of the flow-performance relationship, based on the estimations by Chevalier and Ellison (1997) for the top-performing mutual funds. First, the flow-performance relation is a smooth increasing function given by $f_T = R^{\alpha}_{iT}$, where $\alpha$ denotes the flow elasticity, the elasticity of money flows with respect to manager’s relative performance. Second, the flow-performance relation resembles the payoff profile of a call option – it is flat until a certain performance threshold $\eta$ and then increases with relative performance – given by $f_T = 1_{\{R_{iT} < \eta\}} + R^{\eta}_{iT} 1_{\{R_{iT} \geq \eta\}}$. This specification exhibits a local convexity, as widely documented for mutual funds. The original optimization problem of maximizing the expected value of (5) is equivalent to maximizing the time-$T$ indirect utility function $v_{iT}$, defined as

$$v_{iT} = \max_{\phi_i} E_T \left[ u_i(W_{iT'}) \right]$$

\(^4\)See Goel and Thakor (2005) for a comprehensive review of the literature.

\(^5\)We assume that a manager accepts all the money flows offered to her by investors. In the hedge fund industry, this is not always the case. For example, as discussed in an FT article (FT Wealth Quarterly March 2007 report) with a telling title “Hedge funds: We don’t want your money,” “Many of the industry’s biggest names ... do not need to expand existing funds further and often believe more cash would hurt their returns.” Hence, the money flows interpretation should be applied with caution in the context of hedge funds.
subject to the dynamic budget constraint (1) for \( t \in [T, T'] \), given the time-\( T \) assets value augmented by money flows, \( W_{iT} f_T \). Lemma 1 presents the time-\( T \) indirect utility function for our flow-performance specifications.

**Lemma 1.**

(i) For the flow-performance function \( f_T = R_{iT}^\alpha \), \( \alpha > 0 \), the time-\( T \) indirect utility function of manager \( i \), \( i = 1, 2 \), is given by

\[
v_{iT} = \frac{1}{1 - \gamma_i} \left( W_{iT}^{1-\theta} R_{iT}^{\theta} \right)^{1-\gamma_i};
\]

(ii) for the flow-performance function \( f_T = \mathbb{1}_{\{R_{iT}<\eta\}} + R_{iT}^\alpha \mathbb{1}_{\{R_{iT}\geq\eta\}} \), \( \alpha > 0 \), the time-\( T \) indirect utility function is

\[
v_{iT} = \begin{cases} 
\frac{1}{1-\gamma_i} W_{iT}^{1-\gamma_i} & R_{iT} < \eta \\
\frac{1}{1-\gamma_i} \left( W_{iT}^{1-\theta} R_{iT}^{\theta} \right)^{1-\gamma_i} & R_{iT} \geq \eta,
\end{cases}
\]

where

\[
\theta = \frac{\alpha}{1 + \alpha},
\]

\[
\gamma_i = \bar{\gamma}_i + \alpha(\bar{\gamma}_i - 1),
\]

with the properties that \( \theta \in [0, 1) \), \( \gamma_i > \bar{\gamma}_i \) if and only if \( \bar{\gamma}_i > 1 \).

Lemma 1 provides a rational justification for managers’ having relative performance concerns, and quantifies the link between the shape of the flow-performance relation and the parameters of the manager’s objective function. For the smooth flow-performance function (Lemma 1(i)), we see that the relative performance concerns enter the indirect utility function (6) in a smooth way, with the parameter \( \theta \) capturing the manager’s relative performance bias, the extent to which she biases her objectives towards relative performance concerns. The special case of \( \theta = 0 \) corresponds to a normal manager with no relative performance concerns. From (6), we also observe that the manager’s attitude towards risk changes in the presence of money flows. Indeed, while \( \bar{\gamma}_i \) represents the manager’s intrinsic risk aversion (over terminal wealth \( W_{iT} \)), the parameter \( \gamma_i \) captures her effective risk aversion (over the composite \( W_{iT}^{1-\theta} R_{iT}^{\theta} \)) in the presence of relative performance concerns.\(^6\) Moreover, the manager’s attitude towards risk is increased (\( \gamma_i > \bar{\gamma}_i \)) by the presence of money flows (\( \alpha > 0 \)) for intrinsic risk aversions greater than unity (\( \bar{\gamma}_i > 1 \)), and is decreased for intrinsic risk aversions less than unity. Interestingly, the latter is true even for a convex flow-performance relationship (\( \alpha > 1 \)). So, the interaction of the concave utility function with the convex flow-performance function does not make the manager more risk tolerant, but in fact induces her to effectively behave as more risk averse. The smooth relative performance objective (6) will be investigated in Section 3.

\(^6\)The difference between the two risk aversion parameters stems from the fact that manager \( i \) assesses a gamble \((W_{iT} + \epsilon, W_{iT} - \epsilon)\) differently in cases with and without money inflows. In the latter case, \( \epsilon \) represents an actual change in wealth. In the former case, changing wealth by \( \epsilon \) leads to money in- or outflows and so effectively the manager faces a different gamble.
We also note that the flow-performance specification with a kink (Lemma 1(ii)) exhibits a
local convexity. This in turn leads to local convexities in the resulting objective function (7),
which may induce risk-shifting behavior. In Section 4, we investigate how such risk-shifting
incentives interact as the two managers compete for the money inflows. Finally, we note that for
our subsequent maximization problems to be well-defined, we will assume the smooth indirect
utility function ((6) or region $R_{iT} \geq \eta$ in (7)) to be globally concave. This is true if and only if
$\gamma_i > 1 - 1/(1 + \alpha)$, which always holds for a relatively risk averse manager $\gamma_i > 1$.

While we use Lemma 1 to derive the indirect utility function for a given flow-performance
function $f_T$, we can also address the reverse question. That is, for a given specification of the
objective function, what is the implied $f_T$ that leads to such a specification? For example, let us
consider the specification that has been used in several papers (Browne (2000), van Binsbergen,
Brandt, and Koijen (2007)):

$$v(W) = \frac{1}{1 - \gamma} \left( \frac{W}{X} \right)^{1 - \gamma},$$

(10)

where $X$ can be an exogenous benchmark (van Binsbergen et al.) or the other manager’s horizon
wealth (Browne). The implied flow-performance function in this case is $f(W, X) = 1/X$. That
is, the size of money flows into a fund does not depend on the fund’s own performance. While it
may well be that (10) has other justifications, it does not appear to be consistent with our fund
flows interpretation.

2.3. Nash Equilibrium Policies

In this paper, we appeal to the Nash equilibrium notion to characterize managers’ behavior in
their strategic interaction via relative performance concerns.\(^7\) In what follows, we assume that
managers have common knowledge of the flow-performance relation and each other’s objectives,
in particular attitude towards risk $\gamma_i$. Given our discussion that a strategic interaction is likely to
occur among the very top managers, this assumption seems reasonable. Indeed, such managers
are often in the spotlight attracting a lot of attention from the media and analysts. In order to
define a Nash equilibrium, we first introduce the best response policies. Throughout the paper,
a symbol with a hat $\hat{\cdot}$ denotes an optimal best response quantity, while one with an asterisk $\ast$
denotes an equilibrium quantity.

**Definition 1.** For a given manager 2’s dynamic policy $\phi_2$, manager 1’s best response $\hat{\phi}_1$ is the

\(^7\)As we employ the concept of Nash equilibrium, we assume that the managers act *non-cooperatively*. In
principle, it is possible that other types of behavior, such as herding (Scharfstein and Stein (1990)), can take place.
However, as discussed in Scharfstein and Stein (1990), considerations such as managerial concern for profits and
wages that depend on relative performance, among others, mitigate herding tendencies. Given that we focus on
the very top funds, such considerations are likely to be important. Moreover, the empirical literature largely fails
to find evidence of herding among mutual fund managers (Grinblatt, Titman, and Wermers (1995), Dass, Massa
and Patgiri (2006)).
solution to the following maximization problem:

\[
\max_{\phi_1} \quad E[v_1(W_{1T}, R_{1T})] \\
\text{subject to} \quad dW_{1t} = [r + \phi_1(t)(\mu - r)]W_{1t}dt + \phi_1(t)\sigma W_{1t}d\omega_t.
\] (11)

Similarly, for a given manager 1’s dynamic policy \(\phi_1\), manager 2’s best response \(\hat{\phi}_2\) is the solution to the problem:

\[
\max_{\phi_2} \quad E[v_2(W_{2T}, R_{2T})] \\
\text{subject to} \quad dW_{2t} = [r + \phi_2(t)(\mu - r)]W_{2t}dt + \phi_2(t)\sigma W_{2t}d\omega_t.
\] (12)

Definition 2. A pure-strategy Nash equilibrium is a pair of investment policies \((\phi^*_1, \phi^*_2, t \in [0, T])\) such that:

(i) \(\phi^*_1\) is manager 1’s best response to \(\phi^*_2\),
(ii) \(\phi^*_2\) is manager 2’s best response to \(\phi^*_1\).

In a Nash equilibrium, each manager strategically accounts for the actions of the other manager, and the equilibrium policies of the two managers are mutually consistent. As discussed previously, in our set-up, for a given horizon wealth \(W_{iT}\) satisfying the budget constraint (2) there exists a unique portfolio policy \(\phi_{it}, t \in [0, T]\), replicating it. Hence, for an equilibrium outcome in investment policies \((\phi^*_{1t}, \phi^*_{2t}, t \in [0, T])\), there is always an equivalent outcome in terms of horizon wealth policies \((W^*_{1T}, W^*_{2T})\). We make use of this duality by solving for the equilibrium horizon wealth, and then deriving the corresponding equilibrium investment policies. While we need to impose a certain restriction on the class of admissible dynamic policies in order to use this technique, it is straightforward to prove (proof available from the authors) that the resulting dynamic policies also constitute an equilibrium within the class of strategies without this restriction.\(^8\)

\(^8\)Specifically, when solving for manager 1’s best-response horizon wealth, we assume that manager 2’s horizon wealth is a fixed function of the state variable \(\xi_T\). This corresponds to the restriction that manager 2 chooses her dynamic strategy at time 0 and does not subsequently change it when observing the dynamics of manager 1’s wealth. It can be easily shown that if we start from the equilibrium strategies with this restriction and allow each manager to react to the trades of the other, then neither manager would optimally deviate from her time-0 strategy. A potentially interesting issue is whether there are other equilibria within the class of strategies without the restriction. Addressing this question turns out to be complex, and is beyond the scope of the current paper.
3. Strategic Asset Allocation with Relative Performance Concerns

3.1. Managers’ Objective Functions

In this Section, we investigate a setting in which the objective functions of managers 1 and 2 are given by

\[ v_1(W_{1T}, W_{2T}) = \frac{1}{1 - \gamma_1} \left( W_{1T}^{1-\theta_1} R_{1T}^{\theta_1} \right)^{1-\gamma_1} = \frac{1}{1 - \gamma_1} \left( W_{1T}^{1-\theta_1} (W_{1T}/W_{2T})^{\theta_1} \right)^{1-\gamma_1}, \quad (13) \]

\[ v_2(W_{1T}, W_{2T}) = \frac{1}{1 - \gamma_2} \left( W_{2T}^{1-\theta_2} R_{2T}^{\theta_2} \right)^{1-\gamma_2} = \frac{1}{1 - \gamma_2} \left( W_{2T}^{1-\theta_2} (W_{2T}/W_{1T})^{\theta_2} \right)^{1-\gamma_2}, \quad (14) \]

where \( \gamma_i > 0, \ 0 \leq \theta_i \leq 1, \ i = 1, 2. \) As shown in Lemma 1(i), such objectives may arise in the presence of a smooth flow-performance function \( f_T = R_{1T}. \) These objectives (13)–(14) also have a natural envy interpretation, meaning that a manager gets upset when the other manager’s wealth increases. With the envy interpretation, \( \theta_i \) represents the strength of envy that manager \( i \) feels towards the other manager, and so we allow the managers’ biases, \( \theta_1 \) and \( \theta_2 \), to be possibly different in this Section, and only to be common across managers under the rational interpretation for which \( \theta \) is commonly driven by the flow elasticity \( \alpha. \)

Before proceeding with the formal analysis, we provide some basic intuition regarding how the optimal behavior of manager 1 may be affected by manager 2’s actions in the presence of relative performance concerns of the form (13). For ease of discussion, we focus on the terminal wealth profiles \( W_{iT} \) and adopt the money-flows interpretation. Suppose that manager 2’s horizon wealth \( W_{2T} \) is constant across all states. Then, the objective function of manager 1, and consequently her optimal wealth \( W_{1T} \), are not affected by the presence of relative performance concerns. Suppose now that manager 2’s horizon wealth increases. This has the following two effects on manager 1. First, higher manager 2’s wealth implies lower money flows for manager 1. As a result, manager 1 wants to increase her wealth so as to restore the previous level of flows. Second, higher \( W_{2T} \) also reduces the incremental effect of a unit change in manager 1’s wealth. In other words, manager 2’s wealth serves as a scaling down factor and so higher \( W_{2T} \) makes it costlier for manager 1 to change her wealth. In the pivotal logarithmic case (\( \gamma_1 = 1 \)), the two effects exactly offset each other. For a relatively risk averse manager, \( \gamma_1 > 1 \), the first effect dominates and manager 1 increases her wealth \( W_{1T} \). For a relatively risk tolerant manager, \( \gamma_1 < 1 \), the second effect dominates, whereby manager 1 decreases her wealth.\(^9\) Note that for \( \gamma_1 > 1 \), manager 1’s response essentially means that manager 1 is chasing manager 2. On the other hand, for \( \gamma_1 < 1 \), manager 1 is a contrarian to manager 2.

\(^9\)For clarity purposes, we describe the changes in managers’ wealth as if there is no budget constraint to be satisfied. Given the budget constraint, the discussion remains valid once the wealth changes are understood as being relative to the wealth in other states. So, the precise reasoning would be as follows. For two arbitrary states \( s_1 \) and \( s_2 \), if manager 2’s wealth in state \( s_1 \) is increased relative to her wealth in state \( s_2 \), a relatively risk averse manager 1, \( \gamma_1 > 1 \), responds by increasing her wealth in state \( s_1 \) relative to state \( s_2 \), while a relatively risk tolerant manager 1, \( \gamma_1 < 1 \), responds by decreasing her wealth in state \( s_1 \) relative to state \( s_2 \).
3.2. Equilibrium Investment Policies

In this Section, we determine and provide an explicit characterization of the equilibrium investment policies. First, we present the managers’ best responses in terms of horizon wealth. It is straightforward to show that the best response of manager 1, \( \hat{W}_{1T} \), is given by

\[
\hat{W}_{1T} = (y_1 \xi_T)^{-1/\gamma_1} W_{2T}^{\theta_1(\gamma_1-1)/\gamma_1}.
\]

(15)

where the Lagrange multiplier \( y_1 \) solves \( y_1^{-1/\gamma_1} E \left[ \xi_T^{(\gamma_1-1)/\gamma_1} W_{2T}^{\theta_1(\gamma_1-1)/\gamma_1} \right] = W_{10} \). Similarly, the best response of manager 2, \( \hat{W}_{2T} \), is given by

\[
\hat{W}_{2T} = (y_2 \xi_T)^{-1/\gamma_2} W_{1T}^{\theta_2(\gamma_2-1)/\gamma_2},
\]

(16)

where \( y_2 \) solves \( y_2^{-1/\gamma_2} E \left[ \xi_T^{(\gamma_2-1)/\gamma_2} W_{1T}^{\theta_2(\gamma_2-1)/\gamma_2} \right] = W_{20} \). Focusing on manager 1, her optimal horizon wealth (15) in response to manager 2 is given by the normal component \((y_1 \xi_T)^{-1/\gamma_1}\), the optimal wealth absent relative performance concerns, and a relative performance component \(W_{2T}^{\theta_2(\gamma_2-1)/\gamma_2}\), accounting for relative performance concerns. The relative performance component formalizes the basic intuition offered in the previous Section 3.1. When relatively risk averse \((\gamma_1 > 1)\), manager 1 increases her optimal wealth in response to manager 2’s increasing hers, thereby chasing manager 2. Vice versa, if relatively risk tolerant \((\gamma_1 < 1)\), manager 1 behaves as a contrarian to manager 2 by decreasing her wealth in response to manager 2’s increasing hers.

The next Proposition reports the outcome of the Nash equilibrium in which the managers’ best responses are mutually consistent.10

**Proposition 1.** There exists a unique pure-strategy Nash equilibrium in which the equilibrium investment policies \((\phi_1^*, \phi_2^*)\) are given by

\[
\phi_1^* = \frac{\gamma_2 + \theta_1(\gamma_1-1)}{\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1-1)(\gamma_2-1)} \frac{\kappa}{\sigma},
\]

(17)

\[
\phi_2^* = \frac{\gamma_1 + \theta_2(\gamma_2-1)}{\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1-1)(\gamma_2-1)} \frac{\kappa}{\sigma}.
\]

(18)

The associated equilibrium horizon wealth profiles \((W_{1T}^*, W_{2T}^*)\) are given by

\[
W_{1T}^* = k_{10} W_{10} \xi_T^{\gamma_2 + \theta_1(\gamma_1-1)/\gamma},
\]

(19)

\[
W_{2T}^* = k_{20} W_{20} \xi_T^{\gamma_1 + \theta_2(\gamma_2-1)/\gamma},
\]

(20)

where \( \gamma \equiv \gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1-1)(\gamma_2-1) \) and the constants \( k_{10}, k_{20} \) are presented in the Appendix.

In the special case of manager 2 having no relative performance concerns, \( \theta_2 = 0, \theta_1 \neq 0, \) the

\[10\]We note that our setting admits time-consistent policies, a potential issue in dynamic strategic games. Indeed, the equilibrium investment policies (17)–(18) demonstrate that the two managers have no incentive to deviate at a later date from the policies chosen at the initial date. Moreover, the equilibrium horizon wealth profiles, (19)–(20), obtained for our constant investment opportunities set, as a function of the state variable \( \xi_T \) hold more generally for stochastic investment opportunities when \((r, \mu, \sigma)\) are arbitrary adapted stochastic processes.
equilibrium policies are

\[
\phi_1^* = \frac{\gamma_2 + \theta_1(\gamma_1 - 1) \kappa}{\gamma_1 \gamma_2} = \frac{1}{\gamma_1 \sigma} + \frac{\theta_1(\gamma_1 - 1) \kappa}{\gamma_1 \gamma_2 \sigma},
\]

(21)

\[
\phi_2^* = \frac{1}{\gamma_2 \sigma}.
\]

(22)

Proposition 1 reveals that the equilibrium policy of manager 1 in presence of relative performance concerns, \( \theta_1 \neq 0 \), but with no strategic interaction, \( \theta_2 = 0 \), is comprised of the normal policy absent relative performance concerns, \( \kappa/\gamma_1 \sigma \), plus a relative performance component, \( \theta_1(\gamma_1 - 1)\kappa/(\gamma_1 \gamma_2 \sigma) \). Consistent with our discussion of chasing versus contrarian behavior, the relative component is positive for a relatively risk averse chaser (\( \gamma_1 > 1 \)), thereby increasing her risk exposure, and negative for a risk tolerant contrarian (\( \gamma_1 < 1 \)), thereby decreasing risk exposure as compared to the normal case. Comparing (17) with (21), we see that the adjustment to manager 1’s investment policy due to the strategic interaction is given by the denominator term \( \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1) \), whose sign depends on whether the two managers are more or less risk averse than a logarithmic investor. In other words, whether a manager’s optimal risk exposure is increased or decreased in the presence of strategic interaction depends on whether the managers are chasers or contrarians, as will be elaborated in the next subsection.

### 3.3. Properties of Investment Policies

In this Section, we investigate how changing a manager’s characteristics – attitude towards risk and relative performance bias – affect her own and the other manager’s equilibrium policies. We start with the envy interpretation for which we may treat the risk aversion parameter \( \gamma_i \) and the bias parameter \( \theta_i \) separately, leading to clearer comparative statics. To be consistent with the objective specifications (13)–(14), we still use \( \gamma_i \) to denote manager \( i \)'s risk aversion, while keeping in mind that under the envy interpretation the effective risk aversion coincides with the intrinsic risk aversion.

When discussing the behavior of investment policies, we employ the following reasoning. We envision that the adjustment to the new equilibrium happens not instantaneously but through a stage-by-stage mechanism. In the first stage, manager 1 changes her policy in response to a change in one of her traits (risk aversion or bias), as prescribed by her policy absent strategic interaction (21). In the second stage, because manager 1’s policy has changed, manager 2 finds that her current investment is no longer optimal, and hence, rebalances. The adjustment mechanism continues until neither manager wants to rebalance her investment. The exact nature of a response at each stage depends on whether the manager is a chaser or contrarian, leading to four possible scenarios with different adjustment mechanisms. We describe in more detail the case of both managers being chasers (\( \gamma_1 > 1, \gamma_2 > 1 \)), and then briefly discuss the other cases by similar arguments. Focusing on risk aversions greater than unity is also motivated by the fact that they are more empirically plausible (see Chetty (2006) and references therein).

Corollary 1 presents the effects of changing manager 1’s relative performance bias and attitude
towards risk on the equilibrium policies.\textsuperscript{11}

**Corollary 1.** Assume the envy interpretation of the relative performance concerns. 

(i) Increasing manager 1’s relative performance bias, $\theta_1$, affects the equilibrium portfolio policies of managers 1 and 2, $\phi_1^*$ and $\phi_2^*$, as reported in Table 1. In the special case of both managers’ having only relative performance concerns, $\theta_1 = \theta_2 = 1$, the equilibrium policies are $\phi_1^* = \phi_2^* = \kappa/\sigma$.

<table>
<thead>
<tr>
<th>$\gamma_1 &gt; 1$, $\gamma_2 &gt; 1$</th>
<th>$\gamma_1 &gt; 1$, $\gamma_2 &lt; 1$</th>
<th>$\gamma_1 &lt; 1$, $\gamma_2 &gt; 1$</th>
<th>$\gamma_1 &lt; 1$, $\gamma_2 &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial \phi_1^*/\partial \theta_1$</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\partial \phi_2^*/\partial \theta_1$</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 1.** The effect of manager 1’s relative performance bias, $\theta_1$, on the equilibrium investments $\phi_i^*$, $i = 1, 2$, under the envy interpretation. The parameter $\gamma_i$ denotes manager $i$’s risk aversion.

(ii) Increasing manager 1’s risk aversion, $\gamma_1$, affects the equilibrium policies of managers 1 and 2, $\phi_1^*$ and $\phi_2^*$, as reported in Table 2.

<table>
<thead>
<tr>
<th>$\gamma_2 &gt; 1$</th>
<th>$\gamma_2 &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial \phi_1^*/\partial \gamma_1$</td>
<td>$\gamma_2 &lt; \frac{\theta_1 - \theta_2}{1 - \theta_1 \theta_2}$</td>
</tr>
<tr>
<td>$\partial \phi_2^*/\partial \gamma_1$</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 2.** The effect of manager 1’s risk aversion, $\gamma_1$, on the equilibrium investments $\phi_i^*$, $i = 1, 2$, under the envy interpretation. The parameter $\theta_i$ denotes manager $i$’s relative performance bias.

Table 1 reveals that when the two managers are chasers ($\gamma_1 > 1$, $\gamma_2 > 1$), they both increase their risk exposure in response to an increase in manager 1’s relative performance bias $\theta_1$. In the first stage of the adjustment mechanism, manager 1 increases her risk exposure. The reason is that a higher bias $\theta_1$ makes manager 1 chase manager 2 even stronger, which is achieved by increasing her risk exposure. In the second stage, manager 2 observes that manager 1 has increased her risk exposure and, being also a chaser, responds by increasing her risk exposure. In the third stage, manager 1, having observed a higher risk exposure of manager 2, increases her risk exposure, and so on. We see that the managers’ actions reinforce each other, and so in the new equilibrium with higher $\theta_1$ both managers have higher risk exposures. We also note that as we move from one stage to another, the magnitudes of the adjustments must decrease to ensure convergence to the new equilibrium.

In the case when manager 1 is a chaser and manager 2 a contrarian ($\gamma_1 > 1$, $\gamma_2 < 1$), manager 1 increases her risk exposure, while manager 2 decreases hers as manager 1’s relative performance bias $\theta_1$ increases.

\textsuperscript{11}Although we only alter manager 1’s parameters of the objective function, it is without loss of generality since we can refer to any of the two managers as “manager 1” and the other as “manager 2".
bias increases. The first stage is as in the previous case, but in the second manager 2 acts in a contrarian way, thus decreasing her risk exposure. In the third stage, manager 1 chases manager 2, and so decreases her risk exposure. Iterating forward, we see that each manager adjusts her risk exposure in the opposite direction from her adjustment made two stages ago but with a reduced magnitude, and so the cumulative effect is as in the first two stages, when the largest adjustments take place. In the case when manager 1 is a contrarian and manager 2 is a chaser ($\gamma_1 < 1$, $\gamma_2 > 1$), manager 1 first reduces her risk exposure as her bias $\theta_1$ increases. Manager 2 then chases manager 1 by decreasing her risk exposure, and so both managers decrease their risk exposures as $\theta_1$ is increased. In the final case when both managers are contrarians ($\gamma_1 < 1$, $\gamma_2 < 1$), two subcases arise, which are analogously understood. These subcases arise since one of the managers may now short the stock, which is possible when both managers are relatively risk tolerant (equations (17)–(18)). Comparative statics for a long position are simply reversed for a short position, and hence the two subcases in Table 1.

Corollary 1(i) further reveals that in the extreme case of both managers’ caring only about relative performance, $\theta_1 = \theta_2 = 1$, they follow identical investment policies, those of a logarithmic-utility normal manager, even though they have different attitudes towards risk.\textsuperscript{12} To see this, consider two fully biased logarithmic managers, $\gamma_1 = \gamma_2 = 1$, who would choose identical risk exposures $\kappa/\sigma$. Now suppose that manager 1’s risk aversion, $\gamma_1$, increases. First, this decreases her normal policy absent any bias. Second, with $\gamma_1 > 1$, manager 1 now becomes a chaser, and so increases the risk exposure. It turns out that in the limiting case of $\theta_1 = 1$, these two effects exactly offset each other. Hence, manager 1 follows the same logarithmic investment policy after $\gamma_1$ changes. Similarly, for manager 2 with $\theta_2 = 1$.

Table 2 investigates how manager 1’s attitude towards risk, $\gamma_1$, affects the equilibrium investments. When her risk aversion increases, its effect on manager 1 is determined in the first stage, independent of whether she is a chaser or a contrarian, and so there are fewer cases in Table 2, depending only on whether manager 2 is a chaser or not. When manager 2 is a chaser ($\gamma_2 > 1$), both managers increase their risk exposures as $\gamma_1$ increases. The adjustment mechanism works as follows. As manager 1 becomes more risk averse, in the first stage of the adjustment mechanism she reduces her risk exposure. In the second stage, manager 2 chases manager 1, and so also reduces her risk exposure. When manager 2 is a contrarian ($\gamma_2 < 1$) however, her attitude towards risk also matters in the comparative statics. This is because manager 1’s response in the first stage depends on whether $\gamma_2$ is high or low, with the threshold given by $\gamma_2 < (\theta_1 - \theta_1 \theta_2)/(1 - \theta_1 \theta_2)$. When $\gamma_1$ increases, manager 1’s normal policy always decreases, but the relative performance adjustment is positive. With $\gamma_2$ sufficiently low, the adjustment term dominates the normal policy, and so manager 1 increases her risk exposure in the first stage. Otherwise, the normal policy dominates, and manager 1 decreases her risk exposure. In the second stage, a contrarian manager 2 then adjusts her risk exposure in the opposite direction.\textsuperscript{13}

\textsuperscript{12}The result is also obtained by Browne (2000), whose agents only have relative performance concerns.

\textsuperscript{13}To understand the effect of $\gamma_2$ on manager 1’s best response, recall that there is no relative performance adjustment when manager 2 is fully invested into the riskless asset, i.e., $W_{2T}$ is constant. The larger the deviation from the riskless strategy, i.e., the lower the risk aversion $\gamma_2$, the more important the relative performance adjustment is.
We now investigate the properties of investment policies under the money flows interpretation for which we need to account for the relation between the common relative performance parameter, \( \theta_1 = \theta_2 = \alpha/(1 + \alpha) \), and the effective risk aversion parameter, \( \bar{\gamma}_i = \gamma_i + \alpha(\bar{\gamma}_i - 1) \), from Lemma 1. Since both are driven by the flow elasticity \( \alpha \) and intrinsic risk aversion \( \bar{\gamma}_i \), we carry out our analysis with respect to these two parameters. In the discussion of Tables 1–2, we have provided a detailed description of the economic mechanism underlying the process of adjustment to a new equilibrium. Since the results below follow from similar reasoning, we only highlight the main differences under the current specification. Corollary 2 reports how the flow elasticity \( \alpha \) and manager 1’s attitude towards risk \( \bar{\gamma}_1 \) affect the equilibrium policies of the two managers.

**Corollary 2.** Assume the fund flows interpretation of the relative performance concerns. 
(i) Increasing the flow elasticity \( \alpha \) affects the equilibrium policies of managers 1 and 2, \( \phi_1^* \) and \( \phi_2^* \), as reported in Table 3.

<table>
<thead>
<tr>
<th>( \bar{\gamma}_1 ) &gt; 1, ( \bar{\gamma}_2 &gt; 1 )</th>
<th>( \bar{\gamma}_1 &gt; 1, \bar{\gamma}_2 &lt; 1 )</th>
<th>( \bar{\gamma}_1 &lt; 1, \bar{\gamma}_2 &gt; 1 )</th>
<th>( \bar{\gamma}_1 &lt; 1, \bar{\gamma}_2 &lt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \partial \phi_1^*/\partial \alpha )</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( \partial \phi_2^*/\partial \alpha )</td>
<td>–</td>
<td>+</td>
<td>–</td>
</tr>
</tbody>
</table>

**Table 3.** The effect of the flow elasticity, \( \alpha \), on the equilibrium investments \( \phi_i^* \), \( i = 1, 2 \), under the fund flows interpretation. The parameter \( \bar{\gamma}_i \) denotes manager \( i \)’s intrinsic risk aversion.

(ii) Increasing manager 1’s intrinsic risk aversion \( \bar{\gamma}_1 \) affects the equilibrium policies of managers 1 and 2, \( \phi_1^* \) and \( \phi_2^* \), as reported in Table 4.

<table>
<thead>
<tr>
<th>( \bar{\gamma}_2 &gt; 1 )</th>
<th>( \bar{\gamma}_2 &lt; 1 )</th>
<th>( \bar{\gamma}_2 &lt; \frac{2\alpha}{1+2\alpha} )</th>
<th>( \bar{\gamma}_2 &gt; \frac{2\alpha}{1+2\alpha} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \partial \phi_1^*/\partial \bar{\gamma}_1 )</td>
<td>–</td>
<td>+</td>
<td>–</td>
</tr>
<tr>
<td>( \partial \phi_2^*/\partial \bar{\gamma}_1 )</td>
<td>–</td>
<td>–</td>
<td>+</td>
</tr>
</tbody>
</table>

**Table 4.** The effect of manager 1’s intrinsic risk aversion, \( \bar{\gamma}_1 \), on the equilibrium investments \( \phi_i^* \), \( i = 1, 2 \), under the fund flows interpretation. The parameter \( \alpha \) denotes the flow elasticity.

The comparative statics in Table 3 are considerably different from those in Tables 1–2, perhaps not surprisingly since now the flow elasticity \( \alpha \) affects the biases and the effective risk aversions of both managers. In particular, for the case of both managers being chasers (\( \bar{\gamma}_1 > 1, \bar{\gamma}_2 > 1 \)), we now have that the more risk averse manager increases her risk exposure, while the more risk tolerant manager reduces hers when the flow elasticity increases.\(^{14}\) When the two managers are

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\(^{14}\)If the two managers have the same intrinsic risk aversion (\( \bar{\gamma}_1 = \bar{\gamma}_2 \)), increasing the flow elasticity has no effect.
of different types, one being a chaser and the other a contrarian, they both increase their risk exposures. When the two managers are of different types, one being a chaser and the other a contrarian, they both increase their risk exposures. Finally, when both managers are contrarians ($\bar{\gamma}_1 < 1$, $\bar{\gamma}_2 < 1$), the more risk averse manager decreases her risk exposure, while the more risk tolerant increases hers.

Table 4 looks at the effect of manager 1’s intrinsic attitude towards risk, $\bar{\gamma}_1 < 1$, on the equilibrium investments. The results are very similar to those in Table 2, which considered the effects of the effective attitude towards risk. The reason is that, under the money flows interpretation, increasing the intrinsic risk aversion $\bar{\gamma}_1$ increases the effective risk aversion $\gamma_1$ (equation (9), Lemma 1), and does not affect the relative performance bias. The only difference now is that we substitute $\alpha/(1 + \alpha)$ for the relative performance biases, $\theta_1$ and $\theta_2$.

Our comparative statics results provide mixed support for an argument put forward in Chevalier and Ellison (1997). They assert that the mutual funds that are well ahead of the market have incentives to gamble as they try to become top performers. Under the envy interpretation, in our leading case when both managers are relatively risk averse ($\gamma_1 > 1$, $\gamma_2 > 1$), the two managers indeed increase the riskiness of their investments due to strategic relative performance concerns (Table 1). However, under the fund flow interpretation, which is more consistent with the argument of Chevalier and Ellison, the relatively more risk averse managers change their policies in opposite directions ($\bar{\gamma}_1 > 1$, $\bar{\gamma}_2 > 1$, Table 3). Moreover, while Chevalier and Ellison emphasize the importance of the convexity of the flow-performance relation, in our analysis whether it is convex ($\alpha > 1$) or not ($\alpha \leq 1$) makes no qualitative difference to any of our conclusions.

Remark 1. Multiple fund managers. To investigate the robustness of our results to multiple managers, we have analyzed a setting with three fund managers. The relative performance of each manager $i$, $i = 1, 2, 3$, is now defined as the ratio of manager $i$’s time-$T$ return to the geometric average of the two other managers’ returns. All other assumptions are as in Sections 2–3. In the interest of space, we do not present the full analysis here (available upon request), but only discuss the most pertinent results. We again obtain a unique Nash equilibrium and solve for the managers’ equilibrium policies in closed-form, which are essentially three-manager counterparts of the expressions in Proposition 1. However, the behavior of these equilibrium investment policies is more involved than that in the current Section 3.3 since there are more possible “chaser/contrarian” subcases. Nevertheless, in the leading case when all three managers are chasers the results are very similar to the two-manager framework. Specifically, under the envy interpretation all managers increase their risk exposures. Under the fund flows interpretation, a manager increases her risk exposure if her intrinsic risk aversion is higher than a certain weighted combination of the two other managers’ risk aversions, and decreases otherwise. on the risk exposures since an increase in riskiness due to a higher relative bias is exactly offset by a decrease in riskiness due to a higher effective risk aversion.
4. Strategic Asset Allocation with Asymmetric Relative Performance Concerns

4.1. Managers’ Objective Functions

In this Section, we investigate a setting in which the objective function of manager 1 is given by

\[ v_1(W_{1T}, W_{2T}) = \begin{cases} 
\frac{1}{1-\gamma_1} W_{1T}^{1-\gamma_1} & W_{1T} < \eta W_{2T} \\
\frac{1}{1-\gamma_1} \left( W_{1T}^{1-\theta_1 \left( \frac{W_{1T}}{\eta W_{2T}} \right)^{\theta_1}} \right)^{1-\gamma_1} & W_{1T} \geq \eta W_{2T},
\end{cases} \]  

(23)

where \( \eta \geq 1 \) is the performance threshold, and manager 2’s objective function is as in (23) with subscripts 1 and 2 switched. As shown in Lemma 1(ii), such a form of the objective function arises in the presence of an option-like flow-performance function

\[ f_T = \mathbb{1}_{\{R_{iT} < \eta\}} + \frac{R_{iT}}{\eta} \mathbb{1}_{\{R_{iT} \geq \eta\}}. \]

An alternative, psychological interpretation is that such an objective function may capture the crave for various (non-monetary) benefits that come with the “star” status, such as fame and media attention. Under this interpretation, in order to become a star each manager needs to outperform the opponent by a margin \( \eta > 1 \). Unlike the envy interpretation, a manager driven by the objective function (23) has an asymmetric perception of outperformance and underperformance, whereby only the former affects her status. For example, such an asymmetry can be due to the well-documented fact that people tend to attribute their success to skill and failure to bad luck (Zuckerman (1979)). While under the behavioral interpretation it is possible that manager \( i \)'s relative performance bias \( \theta_i \) varies independently from her effective risk aversion \( \gamma_i \), accounting for this feature would lead to unnecessary complications.\(^{15}\) For this reason, from now on we assume that the managers’ effective risk aversions are related to their relative performance biases as given in (9), where \( \alpha \) is common for the two managers, and adopt the money flows interpretation in our discussion.

To highlight some features of the objective function (23), we plot in Figure 1 its typical shape. From Figure 1, there are three distinct regions of the objective function, depending on the relation between manager 1’s wealth and the threshold level \( \eta W_{2T} \), i.e., her relative horizon performance. When her wealth is above the threshold, the manager gets money flows, and hence we label her as the \textit{winner}. In this case, she is in the region of objectives augmented by relative performance concerns, driven by her effective risk aversion \( \gamma_1 \). When manager 1’s wealth is below the threshold, she does not get money flows, and so is labelled as the \textit{loser}. In this case, the manager finds herself in the region of normal objectives with no relative performance concerns, driven by her intrinsic risk aversion \( \bar{\gamma}_1 \). Finally, when the performance is around the threshold level, the manager is in the region of local convexity. Consequently, there are two main differences from the smooth objective function of Section 3, given in (13). The first is that now both intrinsic

\(^{15}\)Varying manager \( i \)'s relative performance bias \( \theta_i \), separately from her effective risk aversion \( \gamma_i \), makes the objective function discontinuous at the threshold point \( R_{iT} = \eta \). Under certain conditions on parameters, the magnitude of the discontinuity can become large so as to affect the managers’ behavior. Under the money flows interpretation, the continuity of flow-performance relation necessarily implies the continuity of the objective function. Hence, the analysis of the behavioral interpretation is likely to be different, and is left for future research.
and effective risk aversions directly enter the objective specification, and thus have distinct effects on the optimal policies. The second, and more significant, difference is the presence of the local convexity in (23) which, as established in the existing literature, leads to risk shifting behavior. We show that such convexities coupled with strategic interactions can result in multiple equilibria or no equilibrium. Given the novelty of these phenomena, we provide a detailed analysis of the economic conditions under which they may occur in the next Section 4.2. In Section 4.3, we fully characterize the equilibrium policies when an equilibrium (unique or multiple) exists and also investigate the effects of various parameters on the optimal policies.

4.2. Existence and Uniqueness of Equilibrium Policies

As we demonstrate below, the managers’ risk-shifting motives, together with their strategic interaction, may prevent an equilibrium to occur. In this Section, we investigate the underlying economic mechanisms and establish the conditions for the existence and uniqueness of a Nash equilibrium.

First, we determine the managers’ best responses. The managers’ optimization problems are non-standard since their objective functions are not globally concave. Nevertheless, interior solutions turn out to exist since the managers’ risk aversions limit the sizes of their gambles over the locally convex regions. In Proposition 2 we report the managers’ best responses explicitly in closed form.

**Proposition 2.** The best responses of managers 1 and 2 are given by

\[
\hat{W}_{1T} = \begin{cases} 
(y_1 \xi_T)^{-1/\gamma_1} & y_1 \xi_T > b_1(\eta W_{2T}) \quad (\text{loser}) \\
(1 + \alpha)^{1/\gamma_1}(y_1 \xi_T)^{-1/\gamma_1}(\eta W_{2T})^{\theta(\gamma_1 - 1)/\gamma_1} & y_1 \xi_T \leq b_1(\eta W_{2T}) \quad (\text{winner}), 
\end{cases}
\]

\[
\hat{W}_{2T} = \begin{cases} 
(y_2 \xi_T)^{-1/\gamma_2} & y_2 \xi_T > b_2(\eta W_{1T}) \quad (\text{loser}) \\
(1 + \alpha)^{1/\gamma_2}(y_2 \xi_T)^{-1/\gamma_2}(\eta W_{1T})^{\theta(\gamma_2 - 1)/\gamma_2} & y_2 \xi_T \leq b_2(\eta W_{1T}) \quad (\text{winner}), 
\end{cases}
\]

where the boundary functions \( b_i(\cdot) \) are given by

\[
b_i(W) = (1 + \alpha)^{\tilde{\gamma}_i/\theta} \left( \frac{\tilde{\gamma}_i}{\gamma_i} \right)^{\tilde{\gamma}_i \gamma_i / (\gamma_i - \tilde{\gamma}_i)} W^{-\gamma_i(\gamma_i - \tilde{\gamma}_i)}
\]
and \( y_i > 0 \) solves \( E[\xi_T W_{iT}] = W_{i0}, i = 1, 2 \). Moreover, when manager \( i \) is a winner, her associated relative performance \( \hat{R}_{iT} \) is bounded from below by the minimum outperformance margin, \( \bar{\eta}_i \), given by 
\[
\bar{\eta}_i = (1 + \alpha)^{-1/\alpha}(\bar{\gamma}_i/\gamma_i)^{-\alpha/(\gamma_i - \bar{\gamma}_i)}\eta > \eta.
\]
When manager \( i \) is a loser, her relative performance \( \hat{R}_{iT} \) is bounded from above by the maximum underperformance margin, \( \eta_i \), given by 
\[
\eta_i = (1 + \alpha)^{-1/(1+\alpha)}(\bar{\gamma}_i/\gamma_i)^{-\alpha/(\gamma_i - \bar{\gamma}_i)}\eta < \eta.
\]

Focusing on manager 1, she chooses whether to be a winner or a loser depending on the level of the threshold wealth, \( \eta W_{2T} \), relative to the cost of wealth in that state, \( \xi_T \), where the threshold wealth affects manager 1 through the decreasing boundary function \( b_1(\cdot) \). For a relatively low threshold wealth (high manager 2’s wealth or performance threshold), manager 1 optimally becomes a winner, outperforming the threshold \( \eta W_{2T} \), in which case her best response \( (25) \) has the same form as that of the objective augmented by relative concerns in \( (15) \). Otherwise, for a relatively high threshold wealth, manager 1 opts to be a loser, in which case her best response \( (24) \) has the same form as that of a normal policy. An important feature here is that a manager only considers two outcomes: winning or losing, never opting for a “draw” by choosing her relative performance \( \hat{R}_{iT} \) to be equal or close to the threshold \( \eta \). This is due to the convexity of her the objective function around the threshold, inducing her to gamble so as to end up either a winner or a loser, thus avoiding a draw. As presented in Proposition 2, formally, there exists a manager-specific minimum outperformance margin \( \bar{\eta}_i \), greater than \( \eta \), so that a winner’s relative performance can never be below this margin. Similarly, there is a maximum underperformance margin \( \eta_i \), less than \( \eta \), so that a loser’s relative performance can never be above this margin.

To find a Nash equilibrium, we look for mutually consistent best responses of the two managers. Since for a given state of nature, represented by a realization of \( \xi_T \), a manager is either a winner or a loser, there are four possibilities for an equilibrium with two managers. However, both managers cannot be winners as the performance threshold \( \eta \) is greater than 1, and so at most one manager can get money flows. Hence, for each \( \xi_T \) there are three possible Nash equilibrium outcomes, denoted by (manager 1 outcome, manager 2 outcome): (winner, loser), (loser, winner), or (loser, loser). Note that the condition for the (loser, loser) outcome is \( 1/\eta < W_{1T}/W_{2T} < \eta \), which is only possible if \( \eta > 1 \). A Nash equilibrium exists if the three possible outcomes fully cover the interval \((0, +\infty)\), which represents all possible states of the world \( \xi_T \). There are multiple equilibria whenever any two intervals of \( \xi_T \) over which two outcomes occur overlap, meaning that for some \( \xi_T \) the outcome is not uniquely defined.

The existence of an equilibrium, however, is not always possible in this setting. To see how non-existence may arise, assume for ease exposition that the performance threshold \( \eta \) equals one, ruling out the potential (loser, loser) outcome. Consider a situation when manager 1’s minimum outperformance margin \( \bar{\eta}_1 \) is close to one (her relative performance has to exceed \( \bar{\eta}_1 \) for winning (Proposition 2)), while manager 2’s maximum underperformance margin \( \eta_2 \) is much lower than one (her relative performance has to be below \( \eta_2 \) for losing). In this situation, in some states manager 1 may want to outperform manager 2 by just a little, while manager 2, whenever she decides to be a loser, wants to underperform by a lot. Hence, the managers cannot agree on the
winning/losing margin, resulting in the non-existence of equilibrium.

This non-existence issue would be resolved if the two managers had (sufficiently) similar risk aversions, and hence similar performance margins. Indeed, an equilibrium would now exist because the winner’s outperformance margin $\eta_1$ is consistent with the loser’s underperformance margin $\eta_2$. However, if in a certain state of the world the outcome with manager 1 a winner and manager 2 a loser constitutes an equilibrium, the opposite outcome is also likely to be an equilibrium as the managers are (almost) symmetric – in other words, the multiplicity of equilibrium arises. Intuitively, since each manager is aware of her own and the other’s budget constraint, a manager is indifferent between being a winner and a loser in a particular state since choosing a relatively low wealth (being a loser) in one state will allow her to have a relatively high wealth (be a winner) in another state.

Proposition 3 formalizes the discussion above by providing conditions on the model’s parameters for the existence and uniqueness of an equilibrium.

**Proposition 3.**

(i) A unique Nash equilibrium occurs when

$$\eta > \max \left\{ \left( \frac{B}{A} \right)^{1/(2\gamma_1 \gamma_2)}, \left( \frac{C}{A} \right)^{1/(2\gamma_1 \gamma_2)}, \left( \frac{B}{D} \right)^{1/(2\gamma_1 \gamma_2)} \right\};$$

(ii) multiple Nash equilibria occur when

$$\max \left\{ \eta^{1/2} \gamma_2, C \eta^{-2} \gamma_2 (\gamma_1 + \theta (\gamma_1 - 1)) \right\} \leq \min \left\{ B \eta^{-1/2} \gamma_2, D \eta^{1/2} (\gamma_1 + \theta (\gamma_1 - 1)) \right\};$$

(iii) if neither (29) nor (30) is satisfied, there is no Nash equilibrium.

In the above, the constants $A, B, C, D$ are given by

$$A = (1 + \alpha)^{-\gamma_1 \gamma_2/\theta} \left( \frac{\gamma_1}{\gamma_1} \right)^{-\gamma_1 \gamma_2 (\gamma_1 - \gamma_1)} \gamma_1, B = (1 + \alpha)^{\gamma_1 \gamma_2/\theta} \left( \frac{\gamma_2}{\gamma_2} \right)^{\gamma_1 \gamma_2 (\gamma_2 - \gamma_2)} \gamma_1, \quad C = (1 + \alpha)^{\gamma_1 \gamma_2/\theta - \gamma_2} \left( \frac{\gamma_2}{\gamma_2} \right)^{\gamma_1 \gamma_2 (\gamma_2 - \gamma_2)} \gamma_2, D = (1 + \alpha)^{\gamma_1 \gamma_2/\theta} \left( \frac{\gamma_1}{\gamma_1} \right)^{-\gamma_1 \gamma_2 (\gamma_1 - \gamma_1)} \gamma_2.$$  

Proposition 3 reveals that a unique equilibrium occurs when the performance threshold $\eta$ is above a certain critical value.\(^{16,17}\) As discussed, the non-existence arises when a potential equilibrium is destroyed by a loser who, after observing that the winner’s performance is only
Figure 2. Unique Nash equilibrium. The filled area corresponds to the pairs of managers’ intrinsic risk aversions ($\bar{\gamma}_1, \bar{\gamma}_2$) for which the unique pure-strategy Nash equilibrium occurs. The flow elasticity $\alpha$ increases as we go down the plots. The relative performance threshold $\eta$ increases as we go from the left to the right.

slightly higher, increases her performance in an attempt to become the winner. When $\eta$ increases, it is less likely that the loser would try to become the winner, since it would require a larger increase in performance. As for the multiplicity, it occurs when in some states the managers may switch their ranking, with the winner becoming the loser, and vice versa. When the performance threshold is high, both managers optimally choose to be losers in some states of the world. The characteristic feature of the (loser, loser) outcome is that switching rankings has no effect on the managers’ horizon-wealth profiles since the rankings are the same. When $\eta$ is high, the region in which the managers may want to switch their positions lies within the region where both managers are losers, and so the equilibrium is unique. Given the convoluted nature of conditions
for uniqueness and multiplicity, (29) and (30), we investigate when they are likely to be satisfied numerically.

Figure 2 depicts the values of the managers’ intrinsic risk aversions, $\bar{\gamma}_1$ and $\bar{\gamma}_2$, for which the unique equilibrium occurs, i.e. when (29) is satisfied. For a given flow elasticity $\alpha$ and performance threshold $\eta$, the unique equilibrium is likely to exist for higher values of the risk aversions. The reason is that in states of the world when both managers are losers, high risk aversions prevent each manager from gambling that would destroy the equilibrium. As we have just discussed, increasing the performance threshold $\eta$ works against both non-existence and multiplicity, hence the region $(\bar{\gamma}_1, \bar{\gamma}_2)$ where the unique equilibrium exists expands when $\eta$ increases, as seen by comparing the right panels to the left ones in Figure 2. Finally, increasing the flow elasticity $\alpha$ (moving from the top to bottom plots in Figure 2) amplifies the incentives to gamble for both managers, leading to a smaller region of $(\bar{\gamma}_1, \bar{\gamma}_2)$ for which the equilibrium exists.

Figure 3 plots the values of the managers’ intrinsic risk aversions for which multiple equilibria occur, i.e., when (30) is satisfied. When the performance threshold is low – plots (a), (d), (g) on the left – multiple equilibria obtain provided that the risk aversions are not very different, so that the winner’s outperformance margin is consistent with the loser’s underperformance margin. As the flow elasticity increases, moving from plot (a) down to (g), the gambling behavior becomes more pronounced, and so the set of risk aversions for which equilibria obtain shrinks. As the performance threshold $\eta$ increases, we get the unique equilibrium for high risk aversions, as depicted in Figure 2. Hence, we no longer have multiple equilibria in the region of high risk aversion, as evident in Figure 3, plots (c), (f), and (i) on the right. Finally, we note that increasing the flow elasticity $\alpha$ has a larger effect when the performance threshold is low (plots (a), (d), and (g)) than when the threshold is high (plots (c), (f), and (i)). This is explained by recalling that the non-existence arises when both managers are in the convex region and cannot agree on who the winner is. If the threshold $\eta$ is high, only one manager can be in her risk-shifting region in a given state. For example, manager 1 is in her risk-shifting region when her relative performance $R_{1T}$ is close to $\eta$, meaning that the relative performance of manager 2, $R_{2T} = 1/R_{1T}$, is close to $1/\eta$, which is far from her gambling level $\eta$ when $\eta$ is high. As a result, increasing the flow elasticity $\alpha$ has little impact on the existence of equilibrium when $\eta$ is high.

**Remark 2.** Low performance threshold, $\eta < 1$. While for our two interpretations, money flows and behavioral, the assumption that the performance threshold $\eta$ is higher or equal than unity seems justified, it is possible to envision settings in which the threshold is lower than one. For example, Murphy (1999) documents that the prevalent executive compensation contract in the U.S. is the so called 80/120 plan, whereby the manager receives a fixed base salary plus a bonus if her performance exceeds 80% of a pre-specified performance standard. If we take an industry with a few large companies, it may well be that the CEOs of these companies behave strategically, and our framework could be applied once the performance threshold is set at 0.8. It turns out the condition for multiplicity (30) equally applies to this case. However, unlike the case of $\eta > 1$, unique equilibrium is not possible. When $\eta$ drops below a certain value, (30) is no longer satisfied and the equilibrium does not exist. To understand why low performance...
Figure 3. Multiple Nash Equilibria. The filled area corresponds to the pairs of managers’ intrinsic risk aversions \((\bar{\gamma}_1, \bar{\gamma}_2)\) for which multiple Nash equilibria obtain. The flow elasticity \(\alpha\) increases as we go down the plots. The performance threshold \(\eta\) increases as we go from the left to the right.

threshold leads to non-existence, we note that there emerges an additional outcome when both managers are winners. In this (winner, winner) outcome, both managers’ objective functions are augmented by relative concerns, and so each manager’s actions affect the other’s marginal utility. This imposes an additional restriction on the managers’ behavior, as compared to the case \(\eta > 1\), where we always have at least one loser whose marginal utility is not affected by the winner’s actions. This restriction leads to the non-existence of equilibrium.
4.3. Equilibrium Investment Policies and Properties

In this Section, we describe the (possibly multiple) equilibrium policies and investigate their properties. We start with the case of the unique equilibrium that admits a more comprehensive analysis. Proposition 4 fully characterizes the unique investment policies and horizon wealth profiles.

**Proposition 4.** Assume the condition for the existence and uniqueness of a Nash equilibrium \((29)\) is satisfied. The equilibrium investment policy of manager 1, \(\phi^*_1\), is given by

\[
\phi^*_1 = \frac{K}{\sigma W^*_1 t} \left\{ (1 - N(d(\gamma_1, \beta)))y_{1}^{-1/\gamma_1} Z(\gamma_1, t)\xi_t^{-(\gamma_1 + 1)/\gamma_1} \right.
\]

\[ - n(d(\gamma_1, \beta))y_{1}^{-1/\gamma_1} Z(\gamma_1, t)\xi_t^{-(\gamma_1 + 1)/\gamma_1} / (\kappa \sqrt{T - t}) \]

\[ + N(d(\gamma_1, \beta))y_{1}^{-1/\gamma_1} Z(\gamma_1, t)\xi_t^{-(\gamma_1 + 1)/\gamma_1} / (\kappa \sqrt{T - t}) \right\} \]

where \(N(\cdot)\) is the standard-normal cumulative distribution function, \(n(\cdot)\) is the corresponding probability density function, and

\[
y = y_{1}^{-1/\gamma_1} y_{2}^{-\alpha(\gamma_1 - 1)/(\gamma_1 \gamma_2)} \left[ (1 + \alpha)^{-1} \eta^{-\alpha(\gamma_1 - 1)} \right]^{-1/\gamma_1}, \quad \gamma = \frac{\gamma_1 \gamma_2}{\gamma_2 + \alpha(\gamma_1 - 1)},
\]

\[
Z(z, t) = e^{((1 - z)/z)(r + \kappa^2/(2z))(T - t)}, \quad d(x, x) = \frac{\ln(x/\xi_t) + (r + (2 - z)\kappa^2/(2z))(T - t)}{\kappa \sqrt{T - t}},
\]

\[
\beta = \left[ y_{1}^{-\gamma_1 \gamma_2} y_{1}^{-\gamma_1 \gamma_2} (1 + \alpha)^{-\gamma_1 \gamma_2/\gamma_1} (\gamma_1/\gamma_1)^{-\gamma_1 \gamma_2/\gamma_1} (\gamma_1 \gamma_2/\gamma_1) \right]^{1/(\gamma_1 \gamma_2)}
\]

\(y_i > 0\) solves \(E[\xi_t W^*_1 t] = W_0\), and \(W^*_1 t\) is given in the Appendix. The equilibrium portfolio policy of manager 2, \(\phi^*_2\), is as in \((33)\) with subscripts 1 and 2 switched.

The associated equilibrium outcomes and wealth profiles \((W^*_1 T, W^*_2 T)\) at the horizon are as follows.

When \(\xi_T^{-\gamma_1 \gamma_2} \geq yA \eta^{\gamma_1 \gamma_2}\), the managers are in (winner, loser) and

\[
W^*_1 T = c_1 \xi_T^{-\gamma_1 \gamma_2}, \quad W^*_2 T = (y_2 \xi_T)^{-1/\gamma_2}
\]

When \(yB \eta^{-\gamma_1 \gamma_2} \leq \xi_T^{-\gamma_1 \gamma_2} < yA \eta^{\gamma_1 \gamma_2}\), the managers are in (loser, loser) and

\[
W^*_1 T = (y_1 \xi_T)^{-1/\gamma_1}, \quad W^*_2 T = (y_2 \xi_T)^{-1/\gamma_2}
\]

When \(\xi_T^{-\gamma_1 \gamma_2} < yB \eta^{-\gamma_1 \gamma_2}\), the managers are in (loser, winner) and

\[
W^*_1 T = (y_1 \xi_T)^{-1/\gamma_1}, \quad W^*_2 T = c_2 \xi_T^{-\gamma_1 \gamma_2}
\]

where \(A\) and \(B\) are as given in \((31)\) of Proposition 3, \(y = y_2^{-\gamma_1 \gamma_2}\), and

\[
c_1 = y_{1}^{-1/\gamma_1} y_{2}^{-\theta(\gamma_1 - 1)/(\gamma_1 \gamma_2)} (1 + \alpha)^{1/\gamma_1} \eta^{\theta(\gamma_1 - 1)/(\gamma_1 \gamma_2)},
\]

\[
c_2 = y_{2}^{-1/\gamma_2} y_{1}^{-\theta(\gamma_2 - 1)/(\gamma_2 \gamma_1)} (1 + \alpha)^{1/\gamma_2} \eta^{\theta(\gamma_2 - 1)/(\gamma_2 \gamma_1)}
\]

\(25\)
We first look at the managers’ equilibrium horizon wealth. Figure 4 plots the equilibrium (latter part of Proposition 4), as well as the normal wealth profiles, as a function of the state price density $\xi_T$. From Figure 4(a), in good states (low $\xi_T$), the less risk averse manager 2 has a higher equilibrium wealth than manager 1, in line with the normal wealth profiles as depicted in Figure 4(b). In these states, manager 1 is a loser and manager 2 is a winner, getting the money flows. As we move into intermediate states (middle-$\xi_T$ region), manager 2’s relative performance decreases, and after hitting her minimum outperformance margin $\bar{\eta}_2$, jumps down as manager 2 optimally becomes a loser, no longer getting any flows. Finally, as economic conditions deteriorate (high $\xi_T$), manager 1’s relative performance increases, and after reaching the maximum underperformance margin $\eta_1$, it jumps upwards as manager 1 becomes a winner and receives money flows. From the viewpoint of potential fund investors, Figure 4 illustrates the importance of accounting for the managers’ relative performance concerns. The effect is most pronounced in good and bad
states, where the presence of strategic interactions strongly amplifies the difference between the returns on the managers’ portfolios.

To best highlight the managers’ equilibrium risk taking behavior, Figure 5 plots managers’ equilibrium investments (equation (33)) as a function of their own relative performance at time $t$, $R_{1t} = W_{1t}/W_{2t}$, $R_{2t} = W_{2t}/W_{1t}$. When a manager’s relative performance is much lower than the threshold $\eta$, she is likely to be a loser at the horizon. Hence, her behavior is driven by normal objectives and her optimal policy resembles normal behavior. As her relative performance approaches the threshold, the manager enters the convex region of her objective function and so starts to gamble, giving rise to a hump in her investments, as seen in both panels of Figure 5. Each manager gambles in order to avoid a situation where her horizon performance is close to the threshold level, implying that her policy has to be sufficiently different from the other manager’s policy. For this reason, the more risk averse manager 1 optimally gambles by decreasing her stock holding to differentiate herself from the more risk tolerant manager 2, who follows an overall riskier policy. For manager 2, the opposite is true – she increases her stock holding when gambling. Finally, when above the threshold, the manager is affected by the relative performance concerns and so deviates from the normal policy in the direction determined by employing the adjustment mechanism described in Section 3. Specifically, the more risk averse manager 1 increases her risk taking relative to the normal policy, while the more risk tolerant manager 2 decreases hers, consistent with the case of $(\bar{\gamma}_1 > 1, \bar{\gamma}_2 > 1)$ in Table 3.

The advantage of having an analytical solution is that it greatly facilitates the comparative statics analysis. The sensitivities of the equilibrium investment policies to various parameters are illustrated in Figure 6. Across all panels of Figure 6, the managers follow normal-type policies in the deep underperformance region, while in the deep outperformance region the effects are

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18In related work, Basak, Shapiro, and Pavlova (2007) employed numerical methods to describe the optimal policy of a single manager faced with a linear-convex flow-performance relationship, which is similar to ours, and obtained results similar to Figure 5. To make our analysis self-contained, we still discuss the shape of the optimal policies even though much of this parallels the discussion in Basak et al.
Figure 6. Properties of managers’ equilibrium investment policies. The time-$t$ equilibrium investment policies for varying levels of: i) flow elasticity $\alpha \in \{0.1, 0.3, 0.5\}$ – panels (a) and (b); ii) performance threshold $\eta \in \{1.1, 1.15, 1.2\}$ – panels (c) and (d); iii) manager 1’s risk aversion $\gamma_1 \in \{3, 5, 7\}$ – panels (e) and (f). In all panels, dotted lines correspond to the lowest value of the varied parameter, dashed lines to the average value, and solid lines to the highest value. Here, $t = 0.8$ and the other parameters are as in Figure 4.

as presented in Tables 3–4 and can be understood by employing the adjustment mechanism discussed in Section 3. Hence, we focus on the properties of the managers’ investment policies in the gambling region.\footnote{We also do not consider all 4 possible cases, distinguished by each manager’s risk aversion being greater or less than unity, but only discuss the more empirically plausible case of ($\gamma_1 > 1$, $\gamma_2 > 1$). Since the managers’ behavior in the risk-shifting region is not driven by the chasing or contrarian motives, considering the other 3 cases would
From Figures 6(a)–(b), the gambling becomes more pronounced as the flow elasticity \( \alpha \) increases, reflecting the fact that the two managers become more aggressive when the “prize” for becoming the winner increases. Figures 6(c)–(d) demonstrate that a higher performance threshold \( \eta \) shifts the gambling regions to the right. Finally, Figures 6(e)–(f) reveal that a higher manager 1’s risk aversion tempers manager 1’s incentives to gamble, while having almost no effect on manager 2’s risk taking. The magnitude of gambling depends only on the degree of convexity of the manager 2’s objective function around the threshold level \( \eta W_{1T} \), which is not affected by manager 1’s risk aversion.

We now turn to the case of multiple equilibria, which obtains under the multiplicity condition (30). Proposition 5 provides the full characterization of all horizon wealth profiles that can occur in equilibrium.

**Proposition 5.** Assume the condition for the multiplicity of Nash equilibrium (30) is satisfied. The equilibrium outcomes and horizon wealth profiles \( (W_{1T}^*, W_{2T}^*) \) are as follows.

When \( \xi_T^{\gamma_1-\gamma_2} > y \min \{B\eta^{-\gamma_1\gamma_2}, D\eta^{\gamma_1(\gamma_2+\theta(\gamma_2-1))}\} \), the managers are in (winner, loser) and

\[
W_{1T}^* = c_1\xi_T^{-(\gamma_2+\theta(\gamma_1-1))/(\gamma_1\gamma_2)}, \quad W_{2T}^* = (y_2\xi_T)^{-1/\gamma_2}.
\]

When \( \xi_T^{\gamma_1-\gamma_2} < y \max \{A\eta^{-\gamma_1\gamma_2}, C\eta^{-\gamma_2(\gamma_1+\theta(\gamma_1-1))}\} \), the managers are in (loser, winner) and

\[
W_{1T}^* = (y_1\xi_T)^{-1/\gamma_1}, \quad W_{2T}^* = c_2\xi_T^{-(\gamma_1+\theta(\gamma_2-1))/(\gamma_2\gamma_1)}.
\]

When \( y \max \{A\eta^{-\gamma_1\gamma_2}, C\eta^{-\gamma_2(\gamma_1+\theta(\gamma_1-1))}\} < \xi_T^{\gamma_1-\gamma_2} < y \min \{B\eta^{-\gamma_1\gamma_2}, D\eta^{\gamma_1(\gamma_2+\theta(\gamma_2-1))}\} \), both outcomes, (winner, loser) and (loser, winner), can occur in equilibrium. Here, the constants \( A, B, C, D \) are as in Proposition 3, and \( y, c_1, c_2 \) as in Proposition 4.

Proposition 5 reveals that the multiple equilibrium wealth profiles and the unique ones differ only in the intermediate states, middle-\( \xi_T \) region. When the equilibrium is unique, both managers are losers in the middle region, choosing similar horizon wealth profiles (Figure 4). Now, since the performance threshold is relatively low, such wealth profiles are no longer optimal as both managers have incentives to gamble. So, one of the managers must be a winner, implying that her wealth will be considerably higher than the other manager’s wealth, preventing the latter from gambling. As discussed in Section 4.2, in some states the managers are indifferent to switching their rankings. Since now the rankings are different in all states, neither (winner, winner) nor (loser, loser) outcomes being possible, switching rankings leads to multiple equilibria. Unlike the middle region, when \( \xi_T \) is either relatively low or high, the difference between the managers’ wealth levels is substantial in the unique equilibrium (Figure 4). So, even with a low performance threshold \( \eta \) neither of the two managers has incentives to gamble in these regions, leading to the multiple equilibria wealth profiles being the same as the unique-equilibrium ones.

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20 Due to the nature of multiplicity, describing the corresponding investment policies is not straightforward. Loosely, the number of terms in the expressions for optimal policies depend on the number of discontinuities in the horizon wealth profiles, which can be infinite.
Remark 3. Interim Performance. Starting from the study by Brown, Harlow, and Starks (1996), it has become common in both theoretical and empirical works to consider the interim performance as a pertinent factor that affects risk taking incentives in tournament settings. In some theoretical models, the difference in the managers’ interim performance is critical for the existence of a pure-strategy Nash equilibrium (Taylor (2003)). In our analysis, we can easily accommodate this factor in by introducing an interim performance parameter, $R_{10} > 1$, which gives manager 1 an initial advantage. The time-$t$ performance of manager 1 is now given by $R_{10}W_{1t}/W_{10}$. As a result, manager 1 and 2’s relative performances become $R_{10}R_{1t}$ and $R_{2t}/R_{10}$, respectively, where as before $R_{1t} = W_{1t}/W_{2t}$ and $R_{2t} = W_{2t}/W_{1t}$. In Figures 5–6, we simply need to put the new relative performances in place of $R_{1t}$ and $R_{2t}$. Alternatively, if one wants to keep the old ones, the graphs should be appropriately scaled along the x-axis by a factor of $R_{10}$. Figures 2–3 are not affected. The only additional implication of the introduction of interim performance is the possibility of a new equilibrium, in which manager 1 is a winner at the horizon $T$ with certainty, across all states. We do not present this analysis since such an equilibrium can only occur in a knife-edge case when both managers have identical risk aversions.

5. Conclusion

In this paper, we analyze the optimal portfolios of money managers in presence of strategic interactions driven by relative performance concerns. In our first formulation, when the managers care about relative performance at all levels of their performance, we show that a unique Nash equilibrium always obtains and investigate the ensuing investment policies. In the second formulation, when the relative performance concerns only affect the manager whose relative performance is sufficiently high, we discover the possibility of three distinct results: multiple equilibria, unique equilibrium, or no equilibrium at all. When the equilibrium is unique, we analyze the properties of the optimal investment policies. In the other two cases, we elaborate on the underlying economic mechanisms that lead to non-existence or multiplicity, most of which are driven by the risk shifting incentives of the managers.

Given the novelty of our analysis, we believe there are various promising directions for future research. It would be of interest to extend our framework to investigate the possible strategic interactions among CEOs, whose contracts often include a bonus part for high relative performance (Murphy (1999)). Another natural extension of our framework would be to incorporate flow-performance relations where money flows depend on discrete rankings, leading to discontinuities of the managers’ objective functions. Finally, while we assume that money managers have a perfect knowledge of each other’s attitude towards risk and relative performance bias, it would be valuable to consider a more realistic framework where the managers do not have such knowledge but can learn about each other’s traits by observing the investment policies.
Appendix: Proofs

Proof of Lemma 1. Employing martingale methods, given the CRRA preferences (5), manager i’s optimal time-\(T’\) wealth profile \(\hat{W}_{iT'}\) is given by the first order condition
\[
\hat{W}_{iT'} = \left( y_{iT} \xi_{iT'} \right)^{-1/\gamma_i},
\]
where \(y_{iT} > 0\) is the Lagrange multiplier attached to her time-\(T\) static budget constraint \(E_T[\xi_{iT'} \hat{W}_{iT'}] = \xi_{iT} W_{iT} f_T\). The Lagrange multiplier is found by substituting (A1) into the budget constraint, which yields
\[
y_{iT}^{-1/\gamma_i} = \frac{\xi_{iT} W_{iT} f_T}{E_T[\xi_{iT'}^{1-1/\gamma_i}]}.
\]
Plugging (A2) into (A1), we find manager i’s optimal time-\(T'\) wealth profile:
\[
\hat{W}_{iT'} = \frac{\xi_{iT} W_{iT} f_T}{E_T[\xi_{iT'}^{1-1/\gamma_i}]} \xi_{iT'}^{-1/\gamma_i}
\]
Combining (6) and (A3) yields the time-\(T\) indirect utility function
\[
v_{iT} = E_T[u_i(\hat{W}_{iT'})] = \frac{1}{1 - \gamma_i} \left( \xi_{iT} W_{iT} f_T \right)^{1-\gamma_i} \left( \frac{E_T[\xi_{iT'}^{1-1/\gamma_i}]}{E_T[\xi_{iT'}^{1-1/\gamma_i}]} \right)^{1-\gamma_i}
\]
Since \(\xi_t\) follows a geometric Brownian motion with constant drift and volatility, we have that \(E_T[\xi_{iT'}^{1-1/\gamma_i}] = a_1 \xi_{iT}^{-1/\gamma_i}\) and \(E_T[\xi_{iT'}^{1-1/\gamma_i}] = a_2 \xi_{iT}^{-1/\gamma_i}\) where \(a_1\) and \(a_2\) are some constants depending on \(r, \kappa,\) and \(T' - T,\) and we drop them without loss of generality since they do not affect the optimal behavior. Finally, substituting the two expectations into (A4), we get
\[
v_{iT} = \frac{1}{1 - \gamma_i} (W_{iT} f_T)^{1-\gamma_i}
\]
The indirect utility functions (6)–(7) follow by plugging the corresponding flow-performance functions into (A5) after some manipulation. The stated properties of \(\theta\) and \(\gamma_i\) are immediate.

Q. E. D.

Proof of Proposition 1. Substituting manager 2’s best response (16) into manager 1’s best response (15) yields:
\[
\hat{W}_{1T} = (y_1 \xi_T)^{-1/\gamma_1} \left( (y_2 \xi_T)^{-1/\gamma_2} \hat{W}_{1T}^{\theta_2(\gamma_2 - 1)/\gamma_2} \right)^{\theta_1(\gamma_1 - 1)/\gamma_1}
\]
\[
= (y_1 \xi_T)^{-1/\gamma_1} (y_2 \xi_T)^{-\theta_1(\gamma_1 - 1)/(\gamma_1 \gamma_2)} \hat{W}_{1T}^{\theta_2(\gamma_2 - 1)/(\gamma_1 \gamma_2)} ,
\]
from which we deduce
\[
\hat{W}_{1T} = y_1^{-\gamma_2/\gamma_2} y_2^{-\theta_1(\gamma_1 - 1)/\gamma_1} \xi^{-\gamma_1 (\gamma_2 + \theta_1(\gamma_1 - 1))/\gamma_1}
\]
where \(\gamma \equiv \gamma_1 \gamma_2 - \theta_1 \theta_2 (\gamma_1 - 1) (\gamma_2 - 1)\). Substituting (A6) into the static budget constraint \(E[\xi_T \hat{W}_{1T}] = W_{10}\), we get
\[
y_1^{-\gamma_2/\gamma_2} y_2^{-\theta_1(\gamma_1 - 1)/\gamma_1} E \left[ \frac{\xi_T^{1-\gamma_2/\gamma_2 + \theta_1(\gamma_1 - 1)/\gamma_1}}{\gamma_1} \right] = W_{10}
\]
or
\[
y_1^{-\gamma_2/\gamma_2} y_2^{-\theta_1(\gamma_1 - 1)/\gamma_1} = \frac{W_{10}}{E \left[ \frac{\xi_T^{1-\gamma_2/\gamma_2 + \theta_1(\gamma_1 - 1)/\gamma_1}}{\gamma_1} \right]}
\]
Plugging (A7) into (A6) yields:

\[ W_{1T}^* = \frac{W_{10}}{E} \left[ \xi_T^{1-(\gamma_2+\theta_1(\gamma_1-1))/\gamma} \right] \xi_T^{-(\gamma_2+\theta_1(\gamma_1-1))/\gamma}. \]  

(A8)

Since \( \xi_T \) is lognormally distributed with constant mean and variance, one can show that

\[ E_t \left[ \xi_T^{1-(\theta_1(\gamma_1-1)+\gamma_2)/\gamma} \right] = Z(\gamma/(\gamma_2+\theta_1(\gamma_1-1)), t) \xi_t^{1-(\gamma_2+\theta_1(\gamma_1-1))/\gamma}, \]

where \( Z(\cdot) \) is as defined in Proposition 4. As \( \xi_0 = 1 \), we obtain that \( E \left[ \xi_T^{1-(\gamma_2+\theta_1(\gamma_1-1))} \right] = Z(\gamma/(\gamma_2+\theta_1(\gamma_1-1)), 0) \). Denoting the inverse of this quantity by \( k_1 \), we get (17).

To derive the associated equilibrium investment policies, we first determine the time-\( t \) value of the equilibrium wealth \( W_{1t}^* \). Since \( \xi_t W_{1t}^* \) is a martingale, we have

\[ \xi_t W_{1t}^* = E_t[\xi_T W_{1T}^*] = k_1 W_{10} E_t[\xi_T^{1-(\gamma_2+\theta_1(\gamma_1-1))/\gamma}] = k_1 W_{10} Z(\gamma/(\gamma_2+\theta_1(\gamma_1-1)), t) \xi_t^{1-(\gamma_2+\theta_1(\gamma_1-1))/\gamma}. \]

Hence, the optimal horizon wealth \( W_{1t}^* = k_1 W_{10} Z(\gamma/(\gamma_2+\theta_1(\gamma_1-1)), t) \xi_t^{1-(\gamma_2+\theta_1(\gamma_1-1))/\gamma} \). Applying Itô’s Lemma to \( W_{1t}^* \), it is easy to show that its diffusion term is equal to \( \kappa(\gamma_2+\theta_1(\gamma_1-1))/\gamma \). Equating this term to the diffusion term of the investment wealth process \( \phi_t^1 \sigma W_{1t}^* \), as given in (1), yields (17). For manager 2, the derivations are analogous. The special case presented is immediate.

**Proof of Corollary 1.** First, we differentiate manager 1’s investment policy (17) with respect to her bias \( \theta_1 \). From the basic rules of differentiation, the derivative is a ratio in which the denominator as in (17) but squared. The numerator is given by

\[
(\gamma_1 - 1)(\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1)) + (\gamma_2 + \theta_1(\gamma_1 - 1))\theta_2(\gamma_1 - 1)(\gamma_2 - 1) \\
= (\gamma_1 - 1)(\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1) + \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1) + \gamma_2 \theta_2(\gamma_2 - 1)) \\
= \gamma_2(\gamma_1 - 1)(\gamma_1 + \theta_2(\gamma_2 - 1)).
\]

Hence,

\[
\frac{\partial \phi_1^*}{\partial \theta_1} = \frac{\gamma_2(\gamma_1 - 1)(\gamma_1 + \theta_2(\gamma_2 - 1))}{(\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1))^2} \frac{\kappa}{\sigma}.
\]

Differentiating manager 1’s policy with respect to her risk aversion \( \gamma_1 \), we get in the numerator

\[
\theta_1(\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1)) - (\gamma_2 + \theta_1(\gamma_1 - 1))(\gamma_2 - \theta_1 \theta_2(\gamma_2 - 1)) \\
= \theta_1 \gamma_1 \gamma_2 - \theta_1^2 \theta_2(\gamma_1 - 1)(\gamma_2 - 1) - \gamma_2 + \gamma_2 \theta_1 \theta_2(\gamma_2 - 1) + \gamma_2 \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1) \\
= \gamma_2(\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_2 - 1) - \theta_1(\gamma_1 - 1)) = \gamma_2(\theta_1 \gamma_2 - \theta_1 \theta_2(\gamma_2 - 1)).
\]

Hence,

\[
\frac{\partial \phi_1^*}{\partial \gamma_1} = \frac{\gamma_2(\theta_1 \gamma_2 - \theta_1 \theta_2(\gamma_2 - 1))}{(\gamma_1 \gamma_2 - \theta_1 \theta_2(\gamma_1 - 1)(\gamma_2 - 1))^2} \frac{\kappa}{\sigma}.
\]

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Similarly, for manager 2 we have
\[
\frac{\partial \phi_2^*}{\partial \theta_1} = \frac{\theta_2(\gamma_2 - 1)(\gamma_1 - \gamma_2)(\gamma_1 + \theta_2(\gamma_2 - 1)) \kappa}{(\gamma_1 \gamma_2 - \theta_1 \theta_2 (\gamma_1 - 1)(\gamma_2 - 1))^2 \sigma},
\]
\[
\frac{\partial \phi_2^*}{\partial \gamma_1} = \frac{\theta_2(\gamma_2 - 1)(\theta_1 - \gamma_2 + \theta_1 \theta_2(\gamma_2 - 1)) \kappa}{(\gamma_1 \gamma_2 - \theta_1 \theta_2 (\gamma_1 - 1)(\gamma_2 - 1))^2 \sigma}.
\]
The results of Tables 1–2 are obtained by plugging the relevant \( \gamma_1 \) and \( \gamma_2 \) into the above expressions.

**Proof of Corollary 2.** When the flow elasticity changes, the managers’ biases and effective risk aversions change. Because of this, the derivations are more cumbersome. In the interest of space, we present the final results.

\[
\frac{\partial \phi_1^*}{\partial \alpha} = \frac{\gamma_2(\gamma_1 - 1)(\gamma_1 - \gamma_2)}{(\gamma_1 \gamma_2 - \theta_1 \theta_2 (\gamma_1 - 1)(\gamma_2 - 1))^2 \sigma},
\]
\[
\frac{\partial \phi_2^*}{\partial \alpha} = \frac{-\gamma_1(\gamma_2 - 1)(\gamma_1 - \gamma_2)}{(\gamma_1 \gamma_2 - \theta_1 \theta_2 (\gamma_1 - 1)(\gamma_2 - 1))^2 \sigma}.
\]
The results of Tables 3 are obtained by plugging the relevant \( \bar{\gamma}_1 \) and \( \bar{\gamma}_2 \) into the above expressions. Since changing the intrinsic risk aversion \( \gamma_1 \) only affects the effective risk aversion \( \gamma_1 \), the derivatives with respect to \( \bar{\gamma}_1 \) and \( \gamma_1 \) always have the same sign.

**Proof of Proposition 2.** We consider only manager 1; for manager 2 the analysis is analogous. Fixing the horizon wealth profile of manager 2, \( W_{2T}(\xi_T) \), we look for the manager 1’s optimal horizon wealth profile \( \bar{W}_{1T} \). Although manager 1’s objective function has a region of local convexity, we can still use standard optimization techniques once we concavify the objective function (see Basak, Pavlova, and Shapiro (2006) for a more formal proof in a similar setting). Concavification involves finding the range \([W, \bar{W}]\) and the coefficients \( a \) and \( b_1 \) such that replacing \( v_1(\cdot) \) within the range \([W, \bar{W}]\) with a chord \( a + b_1 W_{1T} \) will result in a globally concave objective function. Noting that the chord must be tangent to \( v_1(\cdot) \) at \( W \) and \( \bar{W} \), we have the following system of equations to solve for:

\[
a + b_1 W = \frac{W^{1-\gamma_1}}{1-\gamma_1} \tag{A9}
\]
\[
a + b_1 \bar{W} = \frac{1}{1-\gamma_1} \left( W^{1-\theta} \left( \frac{W}{\bar{W}} \right)^\theta \right)^{1-\gamma_1} \tag{A10}
\]
\[
b_1 = \bar{W}^{-\gamma_1} = (1+\alpha)\bar{W}^{-\gamma_1} W_2^{\theta(\gamma_1-1)}. \tag{A11}
\]

Subtracting (A9) from (A10) yields

\[
b_1(\bar{W} - W) = \frac{1}{1-\gamma_1} \left( W^{1-\gamma_1}(\eta W_2)^{\theta(\gamma_1-1)} - W^{1-\gamma_1} \right). \tag{A12}
\]

Expressing \( W \) and \( \bar{W} \) in terms of \( b_1 \) and \( W_2 \) (from (A11)) and plugging into (A12) gives

\[
b_1(b_1^{-1/\gamma_1}(1+\alpha)^{1/\gamma_1} W_2^{\theta(\gamma_1-1)/\gamma_1} - b_1^{-1/\gamma_1})
\]
\[= \frac{1}{1-\gamma_1} \left( b_1^{-(\gamma_1-1)/\gamma_1} (1+\alpha)^{(\gamma_1-1)/\gamma_1} W_2^{-\theta(\gamma_1-1)^2/\gamma_1} (\eta W_2)^{\theta(\gamma_1-1)} - b_1^{(\gamma_1-1)/\gamma_1} \right),
\]

Q.E.D.
which after some algebra yields the boundary function (28). If \( \xi_T y_1 \) is higher that the slope of the concavification line \( b_1 \), the optimal wealth is to the left from \( W \), i.e., manager 1 chooses to be a loser and her normal-type policy (24) obtains. Otherwise, she becomes a winner and (25) obtains, accounting for relative concerns as in Section 3.

Manager 1’s relative performance is closest to the threshold \( \eta W_{2T} \) when she is indifferent between being a winner and a loser, i.e., when

\[
y_1 \xi_T = b_1(\eta W_{2T}) = (1 + \alpha)^{\gamma_1/\theta} (\gamma_1/\gamma_1)^{\xi_1 \gamma_1 / (\gamma_1 - \gamma_1)} (\eta W_{2T})^{-\gamma_1}.
\]

(A13)

If manager 1 chooses to be a winner, her “minimum outperformance margin” \( \bar{\eta}_1 \) is obtained by plugging (A13) into (25) and dividing the resulting wealth by \( W_{2T} \). This yields

\[
\bar{\eta}_1 = (1 + \alpha)^{1/\gamma_1} \left((1 + \alpha)^{\gamma_1/\theta} (\gamma_1/\gamma_1)^{\xi_1 \gamma_1 / (\gamma_1 - \gamma_1)} (\eta W_{2T})^{-\gamma_1}\right)^{-1/\gamma_1} (\eta W_{2T})^{\theta(\gamma_1 - 1)/\gamma_1 / W_{2T}}
\]

\[
\equiv (1 + \alpha)^{(1 - \gamma_1/\theta)/\gamma_1} (\gamma_1/\gamma_1)^{-\xi_1/\gamma_1} (\eta W_{2T})^{(\xi_1 + \theta(\gamma_1 - 1))/\gamma_1 / W_{2T}}
\]

\[
= (1 + \alpha)^{-1/\alpha} (\gamma_1/\gamma_1)^{-\gamma_1/\gamma_1} \eta,
\]

where in the last equality we use (8)–(9) to simplify the expressions. If manager 1 chooses to be a loser, her “maximum underperformance margin” \( \underline{\eta}_1 \) is obtained by substituting (A13) into (24) and dividing the result by \( W_{2T} \), which yields

\[
\underline{\eta}_1 = \left((1 + \alpha)^{\gamma_1/\theta} (\gamma_1/\gamma_1)^{\xi_1 \gamma_1 / (\gamma_1 - \gamma_1)} (\eta W_{2T})^{-\gamma_1}\right)^{-1/\gamma_1} / W_{2T} = (1 + \alpha)^{-1/\theta} (\gamma_1/\gamma_1)^{-\xi_1/\gamma_1} \eta.
\]

Q.E.D.

**Proof of Proposition 3.** For a given realization of \( \xi_T \), we can have one of the three outcomes: (winner, loser), (loser, winner), or (loser, loser). From the best responses (24)–(27), we can determine the regions of \( \xi_T \) for which these outcomes can occur.

**winner, loser.** From (25), manager 1 chooses to be a winner if \( y_1 \xi_T \leq b_1(\eta W_{2T}) \). Plugging manager 2’s wealth, given by (26) as she is a loser, and using the definition of \( b_1(\cdot) \) (28), yields

\[
y_1 \xi_T \leq (1 + \alpha)^{\gamma_1/\theta} (\gamma_1/\gamma_1)^{\xi_1 \gamma_1 / (\gamma_1 - \gamma_1)} (\xi_T y_2)^{\gamma_1/\gamma_2} \eta^{-\gamma_1}.
\]

Rearranging, we get

\[
\xi_T^{\gamma_1 - \gamma_2} \geq y_1^{\gamma_2} y_2^{-\gamma_1} (1 + \alpha)^{\gamma_1/\theta} (\gamma_1/\gamma_1)^{\xi_1 \gamma_1 / (\gamma_1 - \gamma_1)}\eta^{-\gamma_2} \gamma_1 \gamma_2 \equiv y A y^{\gamma_1 \gamma_2},
\]

(A14)

where \( y = y_1^{\gamma_2} y_2^{-\gamma_1} \) and \( A \) is as given in (31).

From (25), manager 2 chooses to be a loser if \( y_2 \xi_T > b_2(\eta W_{1T}) \). Plugging manager 1’s wealth \( W_{1T} \), given in (25), and expanding \( b_2 \) yields

\[
y_2 \xi_T > (1 + \alpha)^{\gamma_2/\theta}(\gamma_2/\gamma_2)^{\gamma_2 \gamma_2 / (\gamma_2 - \gamma_2)} (\eta(1 + \alpha)^{1/\gamma_1} (y_1 \xi_T)^{-1/\gamma_1} (\eta W_{2T})^{\theta(\gamma_1 - 1)/\gamma_1})^{-\gamma_2}.
\]

After some algebra, we get

\[
\xi_T^{\gamma_1 - \gamma_2} > y_1^{\gamma_2} y_2^{-\gamma_1} (1 + \alpha)^{\gamma_1 \gamma_2 / \theta - \gamma_2} (\gamma_2 / \gamma_2)^{\gamma_2 \gamma_2 / (\gamma_2 - \gamma_2)} \eta^{-\gamma_2(\gamma_1 + \theta(\gamma_1 - 1))} \equiv y C \eta^{-\gamma_2(\gamma_1 + \theta(\gamma_1 - 1))}.
\]

(A15)
where $C$ is as given in (32). The outcome (winner, loser) can occur provided that both (A14) and (A15) are satisfied, which means $\xi_T$ satisfies:

$$\xi_T^{\gamma_1-\gamma_2} > y \max \left[ A \eta^{\gamma_1 \gamma_2}, C \eta^{\gamma_2 (\gamma_1 + \theta (\gamma_1 - 1))} \right].$$  \quad (A16)

**loser, winner.** The expressions are obtained from (A14) and (A15) by switching subscripts 1 and 2, leading to the conditions on $\xi_T^{\gamma_2-\gamma_1}$. For ease of comparison with (winner, loser) case, we then invert the obtained inequalities to get the conditions on $\xi_T^{\gamma_1-\gamma_2}$:

$$\xi_T^{\gamma_1-\gamma_2} \leq y_1 \gamma_2 y_2 \gamma_1 \left(1 + \alpha \right) \gamma_2/\theta \left(\gamma_2/\gamma_1 \right)^{\gamma_2/\gamma_1} \equiv yB \eta^{\gamma_1 \gamma_2},$$  \quad (A17)

$$\xi_T^{\gamma_1-\gamma_2} > y_1 \gamma_2 y_2 \gamma_1 \left(1 + \alpha \right) \gamma_2/\theta \left(\gamma_1/\gamma_2 \right)^{-\gamma_2/\gamma_1} \equiv yD \eta^{\gamma_1 (\gamma_2 + \theta (\gamma_2 - 1))},$$  \quad (A18)

where $B$ and $D$ are given by (31) and (32), respectively. Combining the two conditions, (loser, winner) can occur for $\xi_T$ satisfying

$$\xi_T^{\gamma_1-\gamma_2} < y \min \left[ B \eta^{\gamma_1 \gamma_2}, D \eta^{\gamma_2 (\gamma_2 + \theta (\gamma_2 - 1))} \right].$$  \quad (A19)

**loser, loser.** The conditions for this outcome follow from the observation that manager $i$ wants to be a loser in those states in which she does not want to be a winner. Hence, manager 1 wants to be a loser for $\xi_T$ such that (A14) is not satisfied. Similarly, manager 2 chooses to be a loser when (A17) does not hold. So, (loser, loser) can occur for $\xi_T$ given by

$$yB \eta^{\gamma_1 \gamma_2} < \xi_T^{\gamma_1-\gamma_2} < yA \eta^{\gamma_1 \gamma_2}.$$  \quad (A20)

Inspection of (A16), (A18), and (A19) reveals that if (loser, loser) region is not empty, i.e., if

$$B \eta^{\gamma_1 \gamma_2} < A \eta^{\gamma_1 \gamma_2},$$  \quad (A20)

the three regions can never overlap, meaning that multiple equilibria are not possible. For the unique equilibrium to exist, i.e., for the three regions to fully cover the interval $\left(0, +\infty\right)$, it must be the case that

$$A \eta^{\gamma_1 \gamma_2} \geq C \eta^{\gamma_2 (\gamma_1 + \theta (\gamma_1 - 1))}, B \eta^{\gamma_1 \gamma_2} \leq D \eta^{\gamma_2 (\gamma_2 + \theta (\gamma_2 - 1))},$$  \quad (A21)

in which case the unique equilibrium has the following structure. (winner, loser) occurs for $\xi_T^{\gamma_1-\gamma_2} > yA \eta^{\gamma_1 \gamma_2}$, (loser, loser) for $yB \eta^{\gamma_1 \gamma_2} < \xi_T^{\gamma_1-\gamma_2} < yA \eta^{\gamma_1 \gamma_2}$, and (loser, winner) for $\xi_T^{\gamma_1-\gamma_2} < yB \eta^{\gamma_1 \gamma_2}$. Combining (A20) and (A21) yields the condition for the existence and uniqueness of equilibrium (29).

If (loser, loser) region is empty, i.e., if (A20) is not satisfied, then for an equilibrium to exist it must be the case that the remaining two outcomes fully cover $\left(0, +\infty\right)$. Hence, from (A16) and (A18), we get the multiple equilibria condition (30). In a knife-edge case when (30) holds as an equality, the equilibrium is unique. In this case, (winner, loser) occurs for $\xi_T$ satisfying (A16), (loser, winner) occurs for the other $\xi_T$. In all other cases, when (30) holds
as a strict inequality, multiple equilibria obtains. The structure of the equilibria is as follows. (winner, loser) occurs for \( \xi_T^{1 - \gamma_2} > y \min \left[ B \eta^{1 - \gamma_2}, D \eta^{1 - \gamma_2} \right] \), (loser, winner) occurs for \( \xi_T^{1 - \gamma_2} < y \max \left[ A \eta^{1 - \gamma_2}, C \eta^{1 - \gamma_2} \right] \). The region

\[
y \max \left[ A \eta^{1 - \gamma_2}, C \eta^{1 - \gamma_2} \right] < \xi_T^{1 - \gamma_2} < y \min \left[ B \eta^{1 - \gamma_2}, D \eta^{1 - \gamma_2} \right]
\]

is consistent with both (winner, loser) and (loser, winner) outcomes, hence for such \( \xi_T \) we get the multiplicity of equilibria with these two outcomes. Other structures of equilibrium are not possible, hence there is no pure-strategy Nash equilibrium if neither (29) nor (30) is satisfied.

Q.E.D.

Proof of Proposition 4. From Proposition 3, an equilibrium exists and is unique if the uniqueness condition (29) is satisfied. In the proof of Proposition 3, we have established the structure of the managers’ equilibrium wealth profiles – for each realization of \( \xi_T \) we identified whether manager \( i \), \( i = 1, 2 \), is a winner or a loser. We now determine the associated equilibrium horizon wealth profiles corresponding to these two outcomes. Focusing on manager 1, from (24) her optimal wealth is \( (y_1 \xi_T)^{-1/\gamma_1} \) when she is a loser, i.e., when \( \xi_T^{1 - \gamma_2} \leq y \eta^{1 - \gamma_2} \). Otherwise, when \( \xi_T^{1 - \gamma_2} > y \eta^{1 - \gamma_2} \), manager 1 is a winner, and her best response wealth is given in (25).

As the performance threshold \( \eta \) is greater than one, manager 2 is a loser whenever manager 1 is a winner, and so in equilibrium chooses \( W_{2T}^* = (y_2 \xi_T)^{-1/\gamma_2} \). Plugging this into (25) yields the equilibrium manager 1’s wealth when she is a winner:

\[
W_{1T}^* = y_1^{-1/\gamma_1} y_2^{-\theta(\gamma_1 - 1)/(\gamma_1 \gamma_2)} (1 + \alpha)^{1/\gamma_1} \eta^{\theta(\gamma_1 - 1)/(\gamma_1 \gamma_2)} \xi_T^{-(\gamma_2^{-1} + \theta(\gamma_1 - 1)/(\gamma_1 \gamma_2))}.
\]

The associated equilibrium investment policy of manager 1, \( \phi_{1t}^* \), is derived using the same procedure as described in the proof of Proposition 1. First, we compute manager 1’s time-\( t \) wealth by evaluating the conditional expectation of \( \xi_T W_{1T}^* \) over the relevant regions of \( \xi_T \), obtaining

\[
W_{1t}^* = (1 - N(d(\gamma_1, \beta)))y_1^{-1/\gamma_1} Z(\gamma_1, t) \xi_t^{-1/\gamma_1} + N(d(\bar{y}, \beta)) \bar{y} Z(\gamma_1, t) \xi_t^{-1/\gamma_1},
\]

(A22)

where \( N, d, \beta, Z, \gamma, \bar{y} \) are as defined in Proposition 4. Equating the diffusion term obtained by applying Itô’s Lemma to (A22) with \( \phi_{1t}^* \sigma W_{1t}^* \) yields manager 1’s equilibrium investment policy. For manager 2, the analysis is analogous.

Q.E.D.

Proof of Proposition 5. In Proposition 3 we show that multiple equilibria occur if the multiplicity condition (30) is satisfied. In the proof of Proposition 3, we describe the structure of the multiple equilibria, i.e., the states in which outcome (winner, loser) or outcome (loser, winner) occurs in equilibrium, and the states in which either of the two outcomes is possible. In Proposition 4, we describe what horizon wealth manager \( i \), \( i = 1, 2 \), chooses in equilibrium when she is a winner and when she is a loser. Combining the results of these two Propositions, we obtain the horizon wealth profiles that can occur in the case of multiple equilibria.

Q.E.D.


References


